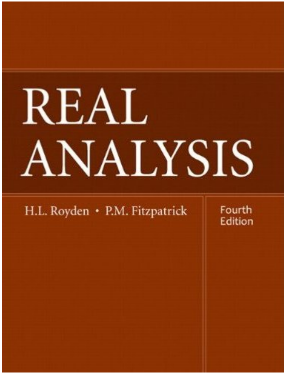
Real Analysis

Chapter 5. Lebesgue Integration: Further Topics

5.3. Characterization of Riemann and Lebesgue Integrability—Proofs of Theorems



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$$\lim_{n\to\infty}\left(\int_E \varphi_n\right) = \int_E f \text{ and } \lim_{n\to\infty}\left(\int_E \psi_n\right) = \int_E f.$$

Proof. For $x \in E$, define $\varphi^*(x) = \lim_{n\to\infty} \varphi_n(x)$ and $\psi^*(x) = \lim_{n\to\infty} \psi_n(x)$.

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Proof. For $x \in E$, define $\varphi^*(x) = \lim_{n\to\infty}\varphi_n(x)$ and $\psi^*(x) = \lim_{n\to\infty}\psi_n(x)$. The functions φ^* and ψ^* are well-defined since $\{\varphi_n\}$ and $\{\psi_n\}$ are monotone and a monotone sequence of extended real numbers converges (to an extended real number). φ^* and ψ^* are measurable by Proposition 3.9.

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Lemma 5.6 (continued 1)

Proof (continued). By monotonicity and properties of limits, $\varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n$ on E and for all n. So $0 \leq \psi^* - \varphi^* \leq \psi_n - \varphi_n$ for all n and by monotonicity of integrals for nonnegative functions (Theorem 4.10) $0 \leq \int_E(\psi^* - \varphi^*) \leq \int_E(\psi_n - \varphi_n)$ for all n, so $0 \leq \int_E(\psi^* - \varphi^*) \leq \lim_{n\to\infty}(\int_E(\psi_n - \varphi_n)) = 0$. Since $\psi^* - \varphi^*$ is nonnegative and measurable and $\int_E(\psi^* - \varphi^*) = 0$, then Proposition 4.9 tells us that $\psi^* = \varphi^*$ a.e. on E. But $\varphi^* \leq f \leq \psi^*$ on E, so $\{\varphi_n\} \to f$ and $\{\psi_n\} \to f$ pointwise a.e. on E. So f is measurable (because $f = \varphi^* = \psi^*$ a.e. on E and φ^*, ψ^* are measurable).

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Proof (continued). By monotonicity and properties of limits, $\varphi_n \le \varphi^* \le f \le \psi^* \le \psi_n$ on E and for all n. So $0 \le \psi^* - \varphi^* \le \psi_n - \varphi_n$ for all n and by monotonicity of integrals for nonnegative functions (Theorem 4.10) $0 \le \int_E (\psi^* - \varphi^*) \le \int_E (\psi_n - \varphi_n)$ for all n, so $0 \le \int_E (\psi^* - \varphi^*) \le \lim_{n\to\infty} (\int_E (\psi_n - \varphi_n)) = 0$. Since $\psi^* - \varphi^*$ is nonnegative and measurable and $\int_E (\psi^* - \varphi^*) = 0$, then Proposition 4.9 tells us that $\psi^* = \varphi^*$ a.e. on E. But $\varphi^* \le f \le \psi^*$ on E, so $\{\varphi_n\} \to f$ and $\{\psi_n\} \to f$ pointwise a.e. on E. So f is measurable (because $f = \varphi^* = \psi^*$ a.e. on E and φ^*, ψ^* are measurable). Next, $0 \le f - \varphi_1 \le \psi_1 - \varphi_1$ on E ($\{\psi_n\}$ is decreasing), and ψ_1 and φ_1 are integrable over E, so by the Integral Comparison Test (Theorem 4.16), f is integrable over E.

Lemma 5.6 (continued 2)

Lemma 5.6. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of functions, each of which is integrable over E, such that $\{\varphi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property $\varphi_n \le f \le \psi_n$ on E for all n. If $\lim_{n\to\infty}\left(\int_E(\psi_n - \varphi_n)\right) = 0$, then $\{\varphi_n\} \to f$ pointwise a.e. on E, $\{\psi_n\} \to f$ pointwise a.e. on E, f is integrable over E,

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Proof (continued). By the above inequalities and monotonicity, $0 \le \int_E \psi_n - \int_E f = \int_E(\psi_n - f) \le \int_E(\psi_n - \varphi_n)$ (also by linearity and the fact that $\{\varphi_n\}$ is increasing) and similarly $0 \le \int_E f - \int_E \varphi_n = \int_E(f - \varphi_n) \le \int_E(\psi_n - \varphi_n)$ ($\{\psi_n\}$ is decreasing), therefore $\lim_{n\to\infty}(\int_E \varphi_n) = \int_E f = \lim_{n\to\infty}(\int_E \psi_n)$ since $\lim_{n\to\infty}\int_E(\psi_n - \varphi_n) = 0$ by hypothesis. $\square$

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Theorem 5.7. Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if f is measurable.

Proof. If f is measurable, then f is Lebesgue integrable by Theorem 4.4.

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$$\underline{\int_E} f = \sup\left\{\int_E \varphi \mid \varphi \text{ is simple, } \varphi \leq f \text{ on } E\right\},$$

$$\overline{\int_E} f = \inf\left\{\int_E \psi \mid \psi \text{ is simple, } \varphi \geq f \text{ on } E\right\},$$

then by the definition of sup and inf, there are sequences of simple functions, $\{\varphi_n\}$ and $\{\psi_n\}$ such that $\varphi_n \leq f \leq \psi_n$ on E for all n, $\underline{\int_E} f = \lim(\int_E \varphi_n)$, and $\overline{\int_E} f = \lim(\int_E \psi_n)$.

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Theorem 5.7 (continued)

Theorem 5.7. Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if f is measurable.

Proof (continued). So

$$\begin{aligned} 0 &= \overline{\int_E} f - \underline{\int_E} f = \lim\left(\int_E \psi_n\right) - \lim\left(\int_E \varphi_n\right) \\ &= \lim\left(\int_E \psi_n - \int_E \varphi_n\right) \\ &= \lim \int_E (\psi_n - \varphi_n) \text{ by linearity, Proposition 4.2.} \end{aligned}$$

Since the max and min of a pair of simple functions are again simple, using the monotonicity of integration and possibly replacing φ_n by $\max_{1\le i\le n}\{\varphi_i\}$ and ψ_n by $\min_{1\le i\le n}\{\psi_n\}$ (pointwise) we may suppose $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing.

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