Chapter 5. Lebesgue Integration: Further Topics
5.3. Characterization of Riemann and Lebesgue Integrability—Proofs of Theorems
1 Lemma 5.6

2 Theorem 5.7
Lemma 5.6. Let \( \{\varphi_n\} \) and \( \{\psi_n\} \) be sequences of functions, each of which is integrable over \( E \), such that \( \{\varphi_n\} \) is increasing while \( \{\psi_n\} \) is decreasing on \( E \). Let the function \( f \) on \( E \) have the property \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \). If \( \lim_{n \to \infty} \left( \int_E (\psi_n - \varphi_n) \right) = 0 \), then \( \{\varphi_n\} \to f \) pointwise a.e. on \( E \), \( \{\psi_n\} \to f \) pointwise a.e. on \( E \), \( f \) is integrable over \( E \),

\[
\lim_{n \to \infty} \left( \int_E \varphi_n \right) = \int_E f \quad \text{and} \quad \lim_{n \to \infty} \left( \int_E \psi_n \right) = \int_E f.
\]

Proof. For \( x \in E \), define \( \varphi^*(x) = \lim_{n \to \infty} \varphi_n(x) \) and \( \psi^*(x) = \lim_{n \to \infty} \psi_n(x) \).
Lemma 5.6. Let \( \{\varphi_n\} \) and \( \{\psi_n\} \) be sequences of functions, each of which is integrable over \( E \), such that \( \{\varphi_n\} \) is increasing while \( \{\psi_n\} \) is decreasing on \( E \). Let the function \( f \) on \( E \) have the property \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \). If \( \lim_{n \to \infty} \left( \int_E (\psi_n - \varphi_n) \right) = 0 \), then \( \{\varphi_n\} \to f \) pointwise a.e. on \( E \), \( \{\psi_n\} \to f \) pointwise a.e. on \( E \), \( f \) is integrable over \( E \),

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\]

Proof. For \( x \in E \), define \( \varphi^*(x) = \lim_{n \to \infty} \varphi_n(x) \) and \( \psi^*(x) = \lim_{n \to \infty} \psi_n(x) \). The functions \( \varphi^* \) and \( \psi^* \) are well-defined since \( \{\varphi_n\} \) and \( \{\psi_n\} \) are monotone and monotone sequence of extended real numbers converge (to an extended real number). \( \varphi^* \) and \( \psi^* \) are measurable by Proposition 3.9.
Lemma 5.6. Let \{\varphi_n\} and \{\psi_n\} be sequences of functions, each of which is integrable over \(E\), such that \{\varphi_n\} is increasing while \{\psi_n\} is decreasing on \(E\). Let the function \(f\) on \(E\) have the property \(\varphi_n \leq f \leq \psi_n\) on \(E\) for all \(n\). If \(\lim_{n \to \infty} \left( \int_E (\psi_n - \varphi_n) \right) = 0\), then \{\varphi_n\} \to f\) pointwise a.e. on \(E\), \{\psi_n\} \to f\) pointwise a.e. on \(E\), \(f\) is integrable over \(E\),

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Proof. For \(x \in E\), define \(\varphi^*(x) = \lim_{n \to \infty} \varphi_n(x)\) and \(\psi^*(x) = \lim_{n \to \infty} \psi_n(x)\). The functions \(\varphi^*\) and \(\psi^*\) are well-defined since \{\varphi_n\} and \{\psi_n\} are monotone and monotone sequence of extended real numbers converge (to an extended real number). \(\varphi^*\) and \(\psi^*\) are measurable by Proposition 3.9.
Lemma 5.6 (continued 1)

Proof (continued). By monotonicity and properties of limits, 
\( \varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n \) on \( E \) and for all \( n \). So \( 0 \leq \psi^* - \varphi^* \leq \psi_n - \varphi_n \) for all \( n \) and by monotonicity of integrals \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \int_E (\psi_n - \varphi_n) \) for all \( n \), so \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \lim_{n \to \infty} (\int_E (\psi_n - \varphi_n)) = 0. \) Since \( \psi^* - \varphi^* \) is nonnegative and measurable and \( \int_E (\psi^* - \varphi^*) = 0 \), then Proposition 4.9 tells us that \( \psi^* = \varphi^* \) a.e. on \( E \). But \( \varphi^* \leq f \leq \psi^* \) on \( E \), so \( \{\varphi^*\} \to f \) and \( \{\psi_n\} \to f \) pointwise a.e. on \( E \). So \( f \) is measurable (because \( f = \varphi^* = \psi^* \) a.e. on \( E \) and \( \varphi^*, \psi^* \) are measurable).
Proof (continued). By monotonicity and properties of limits, 
\( \varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n \) on \( E \) and for all \( n \). So \( 0 \leq \psi^* - \varphi^* \leq \psi_n - \varphi_n \) for all \( n \) and by monotonicity of integrals \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \int_E (\psi_n - \varphi_n) \) for all \( n \), so \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \lim_{n \to \infty} (\int_E (\psi_n - \varphi_n)) = 0 \). Since \( \psi^* - \varphi^* \) is nonnegative and measurable and \( \int_E (\psi^* - \varphi^*) = 0 \), then Proposition 4.9 tells us that \( \psi^* = \varphi^* \) a.e. on \( E \). But \( \varphi^* \leq f \leq \psi^* \) on \( E \), so \( \{ \varphi^* \} \to f \) and \( \{ \psi_n \} \to f \) pointwise a.e. on \( E \). So \( f \) is measurable (because \( f = \varphi^* = \psi^* \) a.e. on \( E \) and \( \varphi^*, \psi^* \) are measurable). Next, \( 0 \leq f - \psi_1 \leq \psi_1 - \varphi_1 \) on \( E \) (\( \{ \psi_n \} \) is decreasing), and \( \psi_1 \) and \( \varphi_1 \) are integrable over \( E \), so by the Integral Comparison Test (Theorem 4.16), \( f \) is integrable over \( E \).
Lemma 5.6 (continued 1)

Proof (continued). By monotonicity and properties of limits, 
\( \varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n \) on \( E \) and for all \( n \). So \( 0 \leq \psi^* - \varphi^* \leq \psi_n - \varphi_n \) for all \( n \) and by monotonicity of integrals \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \int_E (\psi_n - \varphi_n) \) for all \( n \), so \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \lim_{n \to \infty} (\int_E (\psi_n - \varphi_n)) = 0 \). Since \( \psi^* - \varphi^* \) is nonnegative and measurable and \( \int_E (\psi^* - \varphi^*) = 0 \), then Proposition 4.9 tells us that \( \psi^* = \varphi^* \) a.e. on \( E \). But \( \varphi^* \leq f \leq \psi^* \) on \( E \), so \( \{ \varphi^* \} \to f \) and \( \{ \psi_n \} \to f \) pointwise a.e. on \( E \). So \( f \) is measurable (because \( f = \varphi^* = \psi^* \) a.e. on \( E \) and \( \varphi^*, \psi^* \) are measurable). Next, \( 0 \leq f - \varphi_1 \leq \psi_1 - \varphi_1 \) on \( E \) (\( \{ \psi_n \} \) is decreasing), and \( \psi_1 \) and \( \varphi_1 \) are integrable over \( E \), so by the Integral Comparison Test (Theorem 4.16), \( f \) is integrable over \( E \).
Lemma 5.6. Let \( \{\varphi_n\} \) and \( \{\psi_n\} \) be sequences of functions, each of which is integrable over \( E \), such that \( \{\varphi_n\} \) is increasing while \( \{\psi_n\} \) is decreasing on \( E \). Let the function \( f \) on \( E \) have the property \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \). If \( \lim_{n \to \infty} \left( \int_E (\psi_n - \varphi_n) \right) = 0 \), then \( \{\varphi_n\} \to f \) pointwise a.e. on \( E \), \( \{\psi_n\} \to f \) pointwise a.e. on \( E \), \( f \) is integrable over \( E \),

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\lim_{n \to \infty} \left( \int_E \varphi_n \right) = \int_E f \quad \text{and} \quad \lim_{n \to \infty} \left( \int_E \psi_n \right) = \int_E f.
\]

Proof (continued). By the above inequalities and monotonicity,

\[
0 \leq \int_E \psi_n - \int_E f = \int_E (\psi_n - f) \leq \int (\psi_n - \varphi_n) \quad \text{also by linearity and the fact that} \ \{\varphi_n\} \ \text{is increasing}\)
\]

and

\[
0 \leq \int_E f - \int_E \varphi_n = \int_E (f - \varphi_n) \leq \int_E (\psi_n - \varphi_n) \quad \{\psi_n\} \ \text{is decreasing}\)
\]

therefore \( \lim_{n \to \infty} (\int_E \varphi_n) = \int_E f = \lim_{n \to \infty} (\int_E \psi_n) \) since (by the Lebesgue Dominated Convergence Theorem)

\[
\lim_{n \to \infty} \int_E (\psi_n - \varphi_n) = \int_E \lim_{n \to \infty} (\psi_n - \varphi_n) = 0.
\]
Theorem 5.7

**Theorem 5.7.** Let $f$ be a bounded function on a set of finite measure $E$. Then $f$ is Lebesgue integrable over $E$ if and only if $f$ is measurable.

**Proof.** If $f$ is measurable, then $f$ is Lebesgue integrable by Theorem 4.4.
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Suppose $f$ is Lebesgue integrable over $E$ (that is, the upper and lower Lebesgue integrals are equal).
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**Proof.** If \( f \) is measurable, then \( f \) is Lebesgue integrable by Theorem 4.4.

Suppose \( f \) is Lebesgue integrable over \( E \) (that is, the upper and lower Lebesgue integrals are equal). Since

\[
\int_E f = \sup \left\{ \int_E \varphi \mid \varphi \text{ is simple, } \varphi \leq f \text{ on } E \right\},
\]

\[
\int_E f = \inf \left\{ \int_E \psi \mid \psi \text{ is simple, } \varphi \geq f \text{ on } E \right\},
\]

then by the definition of sup and inf, there are sequences of simple functions, \( \{\varphi_n\} \) and \( \{\psi_n\} \) such that \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \), \( \int_E f = \lim(\int_E \varphi_n) \), and \( \int_E f = \lim(\int_E \psi_n) \).
Theorem 5.7. Let \( f \) be a bounded function on a set of finite measure \( E \). Then \( f \) is Lebesgue integrable over \( E \) if and only if \( f \) is measurable.

Proof. If \( f \) is measurable, then \( f \) is Lebesgue integrable by Theorem 4.4.

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then by the definition of sup and inf, there are sequences of simple functions, \( \{ \varphi_n \} \) and \( \{ \psi_n \} \) such that \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \), 

\[
\int_E f = \lim (\int_E \varphi_n), \text{ and } \int_E f = \lim (\int_E \psi_n).
\]
**Theorem 5.7.** Let $f$ be a bounded function on a set of finite measure $E$. Then $f$ is Lebesgue integrable over $E$ if and only if $f$ is measurable.

**Proof (continued).** So

$$0 = \int_E f - \int_E f = \lim \left( \int_E \psi_n \right) - \lim \left( \int_E \varphi_n \right)$$

$$= \lim \left( \int_E \psi_n - \int_E \varphi_n \right)$$

$$= \lim \int_E (\psi_n - \varphi_n) \text{ by linearity.}$$

Since the max and min of a pair of simple functions are again simple, using the monotonicity of integration and possibly replacing $\varphi_n$ by $\max_{1 \leq i \leq n} \{\varphi_i\}$ and $\psi_n$ by $\min_{1 \leq i \leq n} \{\psi_n\}$ (pointwise) we may suppose $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing.
Theorem 5.7 (continued)

**Theorem 5.7.** Let \( f \) be a bounded function on a set of finite measure \( E \). Then \( f \) is Lebesgue integrable over \( E \) if and only if \( f \) is measurable.

**Proof (continued).** So

\[
0 = \int_E f - \int_E f = \lim \left( \int_E \psi_n \right) - \lim \left( \int_E \varphi_n \right)
\]

\[
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\]

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= \lim \int_E (\psi_n - \varphi_n) \text{ by linearity.}
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Theorem 5.7 (continued)

**Theorem 5.7.** Let $f$ be a bounded function on a set of finite measure $E$. Then $f$ is Lebesgue integrable over $E$ if and only if $f$ is measurable.

**Proof (continued).** So

\[
0 = \int_E f - \int_E f = \lim \left( \int_E \psi_n \right) - \lim \left( \int_E \varphi_n \right)
\]

\[
= \lim \left( \int_E \psi_n - \int_E \varphi_n \right)
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= \lim \int_E (\psi_n - \varphi_n) \text{ by linearity.}
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Since the max and min of a pair of simple functions are again simple, using the monotonicity of integration and possibly replacing $\varphi_n$ by $\max_{1 \leq i \leq n}\{\varphi_i\}$ and $\psi_n$ by $\min_{1 \leq i \leq n}\{\psi_n\}$ (pointwise) we may suppose $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing. By Lemma 5.6, $\{\varphi_n\} \to f$ pointwise a.e. on $E$ (so does $\{\psi_n\}$). So $f$ is measurable since it is the pointwise limit a.e. of a sequence of measurable functions (by Proposition 3.9). □