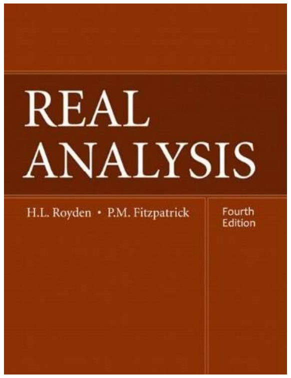
Real Analysis

Chapter 6. Differentiation and Integration

6.1. Continuity and Monotone Functions—Proofs of Theorems



Theorem 6.1

Theorem 6.1. The f be a monotone function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).

Proof. WLOG, say f is monotone increasing. Furthermore, assume (a, b) is bounded (that is, a and b are finite) and f is increasing on the closed interval $[a, b]$. Otherwise, express (a, b) as the union of an ascending sequence of open, bounded intervals, the closures of which are contained in (a, b) (which can be done if a and b are finite with a $\pm 1/n$ approach, or with a $\pm n$ approach if a and/or b is infinite). Then take the unions of the discontinuities in each of this countable collection of intervals.

For each $x_0 \in (a, b)$, f has a finite limit from the left and from the right at a_0. Define

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup\{f(x) \mid a < x < x_0\},$$

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = \inf\{f(x) \mid x_0 < x < b\}.$$

Theorem 6.1 (continued)

Theorem 6.1. The f be a monotone function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).

Proof (continued). Since f is increasing, then $f(x_0^-) \le f(x_0^+)$. So f has a discontinuity at x_0 if and only if $f(x_0^-) < f(x_0^+)$, in which case there is a jump discontinuity at x_0. Define the "jump" interval $J(x_0) = \{y \mid f(x_0^-) < y < f(x_0^+)\}$. Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is (pairwise) disjoint. Therefore, for each $n \in \mathbb{N}$, there are only finitely many jump intervals of length greater than $1/n$. Thus the set of points of discontinuity of f is the union of a countable collection of finite sets and therefore is countable. (An alternative approach is to observe that we can pick a rational number from each jump interval. Different discontinuities are then associated with different rational numbers. Of course, a subset of $\mathbb{Q}$ is countable.) □

Proposition 6.2

Proposition 6.2. Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at the points in $(a, b) \setminus C$.

Proof. The proof is easy for finite C, so WLOG suppose C is countably infinite. Let $\{q_n\}_{n=1}^\infty$ be an enumeration of C. Define function f on (a, b) as $f(x) = \sum_{\{n \mid q_n \le x\}} 1/2^n$ (where $z \in (a, b)$). Notice that $f(x)$ is part of a geometric series which converges to 1, and so $f(x)$ is well-defined. (If a and b are finite, we could extend f to the endpoints as $f(a) = 0$ and $f(b) = 1$; this is Exercise 6.1). Moreover, if $a < u < v < b$ then

$$f(v) - f(u) = \sum_{\{n \mid u < q_n < v\}} \frac{1}{2^n} \ge 0. \qquad (1)$$

Thus f is increasing. Let $x_0 = q_k \in C$. Then by (1), $f(x_0) - f(x) \ge 1/2^k$ for all $x < x_0$ (since $1/2^k$ is included in the sum for $f(x_0)$ but not included in the sum for $f(x)$ where $x < x_0$). Therefore, f is not continuous at x_0 since $\lim_{x \to x_0^-} f(x) \le f(x_0) - 1/2^k$.

Proposition 6.2 (continued)

Proposition 6.2. Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at the points in $(a, b) \setminus C$.

Proof (continued). Now let $x_0 \in (a, b) \setminus C$. Let $n \in \mathbb{N}$. There is an open interval I containing x_0 for which q_n does not belong to I for $1 \leq k \leq n$. So from (1) we have for $x \in I$:

$$|f(x) - f(x_0)| < \sum_{k=n+1}^{\infty} = \frac{1/2^{n+1}}{1 - 1/2} = \frac{1}{2^n}.$$

So for any $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $1/2^n < \varepsilon$. Then pick δ such that $I = (x_0 - \delta, x_0 + \delta)$ contains none of $q_1, q_2, \ldots, q_n$. Then for $x \in I$ we have $|f(x) - f(x_0)| < \varepsilon$. So f is continuous at $x_0 \in (a, b) \setminus C$. □