Chapter 6. Differentiation and Integration
6.1. Continuity and Monotone Functions—Proofs of Theorems
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For each \(x_0 \in (a, b)\), \( f \) has a finite limit from the left and from the right at \(a_0\). Define

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 f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup\{f(x) \mid a < x < x_0\},
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 f(x_0^+) = \lim_{x \to x_0^+} f(x) = \inf\{f(x) \mid x_0 < x < b\}.
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Proof (continued). Since $f$ is increasing, then $f(x_0^-) \leq f(x_0^+)$. So $f$ has a discontinuity at $x_0$ if and only if $f(x_0^-) < f(x_0^+)$, in which case there is a jump discontinuity at $x_0$. Define the “jump” interval $J(x_0) = \{y \mid f(x_0^-) < y < f(x_0^+)\}$. Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is (pairwise) disjoint.
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Proposition 6.2

\textbf{Proposition 6.2.} Let $C$ be a countable subset of the open interval $(a, b)$. Then there is an increasing function on $(a, b)$ that is continuous only at the points in $(a, b) \setminus C$.

\textbf{Proof.} The proof is easy for finite $C$, so WLOG suppose $C$ is countably infinite. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of $C$. 

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$$f(v) - f(u) = \sum_{\{n|u < q_n < v\}} \frac{1}{2^n} \geq 0. \quad (1)$$

Thus $f$ is increasing.
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So for any $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $1/2^n < \varepsilon$. Then pick $\delta$ such that $I = (x_0 - \delta, x_0 + \delta)$ contains none of $q_1, q_2, \ldots, q_n$. Then for $x \in I$ we have $|f(x) - f(x_0)| < \varepsilon$. 
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