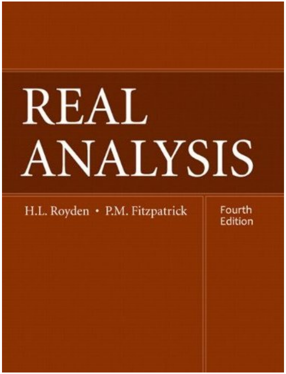
Real Analysis

Chapter 6. Differentiation and Integration

6.1. Continuity and Monotone Functions—Proofs of Theorems



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For each $x_0 \in (a, b)$, f has a finite limit from the left and from the right at a_0. Define

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup\{f(x) \mid a < x < x_0\},$$

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = \inf\{f(x) \mid x_0 < x < b\}.$$

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Proof (continued). Since f is increasing, then $f(x_0^-) \leq f(x_0^+)$. So f has a discontinuity at x_0 if and only if $f(x_0^-) < f(x_0^+)$, in which case there is a jump discontinuity at x_0. Define the "jump" interval $J(x_0) = \{y \mid f(x_0^-) < y < f(x_0^+)\}$. Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is (pairwise) disjoint.

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Proposition 6.2

Proposition 6.2. Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at the points in $(a, b) \setminus C$.

Proof. The proof is easy for finite C, so WLOG suppose C is countably infinite. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of C.

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$$|f(x) - f(x_0)| < \sum_{k=n+1}^{\infty} = \frac{1/2^{n+1}}{1 - 1/2} = \frac{1}{2^n}.$$

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