for each \( n \in \mathbb{N} \).

(6) \( \varnothing = (\mathbb{I} \in (\gamma, \gamma)] \cup \text{and } \forall \varnothing \subseteq \mathbb{I} \in F \), \( \forall I \in \mathbb{I} \), \( \exists \varnothing \subseteq \mathbb{I} \in F \).

If there is a finite disjoint subcollection of \( F \) that covers \( E \), then

(4) \( \{E = (\mathbb{I} \in (\gamma, \gamma)] \cup \text{and } \exists \varnothing \subseteq \mathbb{I} \in F \} = F \).

We therefore have:

since \( F \) is a partition of \( E \), \( \mathbb{I} \subseteq \gamma \) is a subset of the \( \gamma \)-neighborhood of \( \mathbb{I} \) to which contains \( \gamma \). This is \( \mathbb{I} \subseteq \gamma \).

Therefore, \( \forall \mathbb{I} \subseteq \gamma \) is a \( \gamma \)-neighborhood of \( \mathbb{I} \) which contains \( \gamma \). So each element of \( \gamma \) contains no other element of \( \gamma \).

\( \{E = (\mathbb{I} \in (\gamma, \gamma)] \cup \text{and } \exists \varnothing \subseteq \mathbb{I} \in F \} = F \).

\( \{E = (\mathbb{I} \in (\gamma, \gamma)] \cup \text{and } \exists \varnothing \subseteq \mathbb{I} \in F \} = F \).

The Vitali Covering Lemma (continued 2)

The Vitali Covering Lemma (continued 1)
Then by finite subadditivity

\[
\left( \bigcup_{\nu} [a, b] \right) \cap \left( \bigcup_{\mu} [a, b] \right) \subseteq \bigcup_{\nu, \mu} [a, b]
\]

any \( q \in \mathbb{Q} \). Since \( \mathbb{Q} \) is a covering of \( \mathbb{R} \), \( \mathbb{Q} \) can be made arbitrarily small in length. So \( \mathbb{Q} \) is a covering of \( \mathbb{R} \), for \( q \in \mathbb{Q} \).

Lemma 6.3. Let \( f \) be an increasing function on the closed, bounded

\[\text{interval } [a, b]. \text{ Then for each } a < b, \text{ the distance from the midpoint of } [a, b] \text{ to } \mathbb{Q} \text{ is at most } \rho.\]

This proves (2).

Proof of (2).

The Vitali Covering Lemma 6.2 (continued 1)

Lemma 6.3 (continued 1)

Proof (continued 1).

\[\text{Lemma 6.3 (continued 2)}\]

Proof (continued 2).
Lemma 6.3 (continued 2)

**Proof (continued).** However, the function $f$ is increasing on $[a, b]$ and $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of subintervals of $[a, b]$, so

$$\sum_{k=1}^n [f(d_k) - f(c_k)] = (f(d_1) - f(c_1)) + (f(d_2) - f(c_2)) + \cdots$$

$$+(f(d_{n-1}) - f(c_{n-1})) + (f(d_n) - f(c_n))$$

$$= -f(c_1) + (f(d_1) - f(c_2)) + \cdots + (f(d_{n-1}) - f(c_n)) + f(d_n)$$

$$\leq -f(c_1) + 0 + 0 + \cdots + 0 + f(d_n)$$

(where we index the $c_k, d_k$'s in increasing order; because $f$ is an increasing function)

Thus, for each $\varepsilon > 0$ and $\alpha' \in (0, \alpha)$ we have

$$m^*(E_{\alpha}) \leq (1/\alpha') [f(b) - f(a)] + \varepsilon,$$

and so (7) holds.

---

Lebesgue Theorem

**Lebesgue Theorem.**

If the function $f$ is monotonic on the open interval $(a, b)$, then it is differentiable almost everywhere on $(a, b)$.

**Proof.** Let $f$ be increasing on $(a, b)$. WLOG, we take $f$ to be bounded (otherwise we express $(a, b)$ as the union of an ascending sequence of open, bounded intervals on which $f$ is increasing and bounded, show that $f$ is differentiable a.e. on each of these intervals, and then apply continuity of Lebesgue measure (theorem 2.15) to the sequence of measure zero subsets of the ascending sequence of intervals and again conclude that $f$ is a.e. differentiable). The set of points $x \in (a, b)$ at which

$$\overline{D}[f(x)] > D[f(x)]$$

is the union of the sets

$$E_{\alpha, \beta} = \{x \in (a, b) \mid \overline{D}[f(x)] > \alpha > \beta > D[f(x)]\}$$

where $\alpha, \beta \in \mathbb{Q}$.

---

Lemma 6.3 (continued 3)

**Lemma 6.3.** Let $f$ be an increasing function on the closed, bounded interval $[a, b]$. Then for each $\alpha > 0$,

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] \geq \alpha\} \leq \frac{1}{\alpha} [f(b) - f(a)]$$

(7)

and

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\} = 0.$$  (8)

**Proof (continued).** For all $n \in \mathbb{N}$ we have that

$$\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\} \subseteq E_n$$

and therefore by monotonicity of $m^*$ and (7) we have

$$m^*(\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\}) \leq m^*(E_n) \leq \frac{1}{n} [f(b) - f(a)],$$

and so $m^*(\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\}) = 0$ and (8) holds.

---

Lebesgue Theorem (continued 1)

**Proof (continued).** Fix $\alpha, \beta \in \mathbb{Q}$ with $\alpha > \beta$ and set $E = E_{\alpha, \beta}$. Let $\varepsilon > 0$. Choose an open set $O$ for which

$$E \subseteq O \subseteq (a, b) \text{ and } m(O) < m^*(E) + \varepsilon$$

(10)

(which can be done by the infimum definition of outer measure). Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c, d]$ contained in $O$ for which $f(d) - f(c) < \beta(d - c)$. Since $D[f(x)] < \beta$ on $E$, then as explained in the proof of Lemma 6.3, $\mathcal{F}$ is a Vitali covering of $E$. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{(c_k, d_k)\}_{k=1}^n$ of $\mathcal{F}$ for which $m^*[E \setminus (\bigcup_{k=1}^n [c_k, d_k])] < \varepsilon$. By the choice of the intervals $[c_k, d_k]$, we have

$$\sum_{k=1}^n [f(d_k) - f(c_k)] < \beta \sum_{k=1}^n (d_k - c_k)$$

$$\leq \beta m(O) \text{ by monotonicity of } m$$

$$\leq \beta [m^*(E) + \varepsilon] \text{ by (10).}$$

(12)
\[
\frac{1}{n} \sum_{j=1}^{n} \left( \int_{y+j}^{y+n} f(x) \, dx \right) - \frac{1}{n} \sum_{j=1}^{n} \left( \int_{y}^{y+n} f(x) \, dx \right) = 0
\]

This holds for \( n \) sufficiently large. By the Lebesgue differentiation theorem, for almost every \( x \) in \( [a, b] \):

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \int_{y+j}^{y+n} f(x) \, dx \right) = f(x)
\]

for almost every \( x \) in \( [a, b] \).

**Proof.** By Fatou's lemma, we have

\[
\frac{1}{n} \sum_{j=1}^{n} \left( \int_{y+j}^{y+n} f(x) \, dx \right) \leq \int_{y}^{y+n} f(x) \, dx
\]

which converges to \( f(x) \) for almost every \( x \) in \( [a, b] \). Therefore, the sequence of non-negative functions \( \frac{1}{n} \sum_{j=1}^{n} \left( \int_{y+j}^{y+n} f(x) \, dx \right) \) converges pointwise to \( f(x) \).

**Corollary 6.4.** Let \( f \) be an increasing function on the closed, bounded interval \([a, b]\). Then \( f \) is measurable.

**Proof (continued).** Notice that for any \( a, b \) in \([a, b]\) and \( \varepsilon > 0 \),

\[
(\varepsilon, f) - (\varepsilon, f) \leq \int_{a}^{b} f(x) \, dx
\]

for all \( \varepsilon > 0 \).

Therefore, since \( f \) is increasing on \([a, b]\), we have

\[
\int_{a}^{b} f(x) \, dx = \int_{\varepsilon}^{b} f(x) \, dx
\]

for all \( \varepsilon > 0 \).

**Lebesgue Theorem (continued).**

We infer from (12) and (13) that

\[
\varepsilon + \frac{\varepsilon}{g} + (\varepsilon, \omega) \, g = \varepsilon + \left( \frac{\varepsilon + (\varepsilon, \omega) \, g}{I} \right) \geq (\varepsilon, \omega)
\]

for all \( \varepsilon > 0 \).

Therefore, since \( f \) is measurable, it is measurable on the closed, bounded interval \([a, b] \).

**Corollary 6.4.** Let \( f \) be an increasing function on the closed, bounded interval \([a, b]\). Then \( f \) is measurable.

**Proof (continued).** Since \( f \) is increasing on \([a, b]\), we have

\[
(\varepsilon, f) - (\varepsilon, f) \leq \int_{a}^{b} f(x) \, dx
\]

for all \( \varepsilon > 0 \).

Therefore, since \( f \) is measurable, it is measurable on the closed, bounded interval \([a, b]\).

**Lebesgue Theorem (continued).**
\[ (e) f - (q) f \leq \left( \left[ f \right]_{u/t}^{e} \right)_{q}^{\infty} \limsup_{n \to \infty} \leq \]

\[ \left( \left[ f \right]_{u/t}^{e} \right)_{q}^{\infty} \liminf_{n \to \infty} \leq \int_{q}^{e} \]

Combining (16) and (17) gives

\[ (e) f - (q) f \leq \left( \left[ f \right]_{u/t}^{e} \right)_{q}^{\infty} \limsup_{n \to \infty} \]

Thus

\[ [q \cdot e] \quad \text{since} \quad f \quad \text{is increasing on} \quad [u/t + q \cdot e] \quad \text{on} \quad (x) f \equiv (x) f \]

\[ \int_{u/t + e}^{e} \frac{u/t}{1} - (q) f = \]

\[ (x) f \quad \int_{u/t + e}^{e} \frac{u/t}{1} - (x) f \quad \int_{u/t + q \cdot e}^{q \cdot u/t} = \left[ f \right]_{u/t}^{e} \]

Proof (continued).

Corollary 6.4 (continued 2)