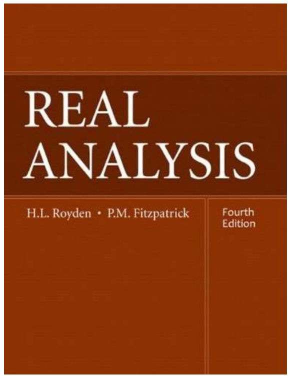
Real Analysis

Chapter 6. Differentiation and Integration

6.2. Differentiability of Monotone Functions: Lebesgue's Theorem—Proofs of Theorems



The Vitali Covering Lemma

The Vitali Covering Lemma.
Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\varepsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ for which

$$m^*\left[E \setminus \bigcup_{k=1}^n I_k\right] < \varepsilon. \qquad (2)$$

Proof. Since $m^*(E) < \infty$, there is an open set $\mathcal{O}$ containing E for which $m(\mathcal{O}) < \infty$ (by the definition of outer measure). Because $\mathcal{F}$ is a Vitali covering of E, then every $x \in E$ is in some interval of length less than $\varepsilon > 0$ (for arbitrary $\varepsilon > 0$). So WLOG we can suppose each interval in F is contained in $\mathcal{O}$. By the countable additivity of measure and monotonicity of measure, if $\{I_k\}_{k=1}^\infty \subseteq \mathcal{F}$ is a collection of disjoint intervals in $\mathcal{F}$ the

$$\sum_{k=1}^\infty \ell(I_k) \le m(\mathcal{O}) < \infty. \qquad (3)$$

The Vitali Covering Lemma (continued 1)

Proof (continued). For $\{I_k\}_{k=1}^\infty \subset \mathcal{F}$, define $\mathcal{F}_n = \{I \in \mathcal{F} \mid I \cap (\cup_{k=1}^n I_k) = \varnothing\}$. So each element of $\mathcal{F}_n$ contains no elements of $\cup_{k=1}^n I_k$. For any $x \in E \setminus (\cup_{k=1}^n I_k)$ (a set open with respect to E), there is an ε-neighborhood of x which does not intersect $\cup_{k=1}^n I_k$. Since $\mathcal{F}$ is a covering of E in the sense of Vitali, then there is $I \in \mathcal{F}$ a subset of the ε-neighborhood of x which contains x. This $I \in \mathcal{F}_n$ since it does not intersect $\cup_{k=1}^n I_k$. We therefore have:

$$\text{if } \{I_k\}_{k=1}^m \subseteq \mathcal{F}, \text{ then } E \setminus (\cup_{k=1}^\infty I_k) \subseteq E \setminus (\cup_{k=1}^n I_k) \subseteq \cup_{i \in \mathcal{F}_n} I$$

$$\text{where } \mathcal{F}_n = \{I \in \mathcal{F} \mid I \cap (\cup_{k=1}^n I_k) = \varnothing\}. \qquad (4)$$

If there is a finite disjoint subcollection of $\mathcal{F}$ that covers E, then $m^*(E \setminus \cup_{k=1}^n I_k) = 0$ and we are done.

The Vitali Covering Lemma (continued 2)

Proof (continued). Otherwise, we inductively choose a disjoint subcollection $\{I_k\}_{k=1}^\infty$ of $\mathcal{F}$ which has the following property:

$$E \setminus \cup_{k=1}^n I_k \subseteq \cup_{k=n+1}^\infty 5 * I_k \text{ for all } n \in \mathbb{N} \qquad (5)$$

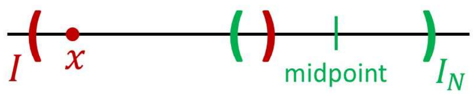
where, for a closed, bounded interval I, "$5 * I$" denotes the closed interval that has the same midpoint as I and 5 times its length.
Let I_1 be any interval in $\mathcal{F}$. Suppose $n \in \mathbb{N}$ and that the finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ has been chosen. Since $E \setminus (\cup_{k=1}^n I_k) \neq \varnothing$ (since we've assumed there is no finite cover of E in this case), the collection $\mathcal{F}_n$ defined in (4) is nonempty. Moreover, the supremum, s_n, of the lengths of the intervals in $\mathcal{F}_n$ is finite since $m(\mathcal{O}) < \infty$ is an upper bound for these lengths. Choose I_{n+1} to be an interval in $\mathcal{F}_n$ for which $\ell(I_{n+1}) > s_n/2$. Notice that $I_{n+1} \in \mathcal{F}_n$ implies $\{I_k\}_{k=1}^\infty$ is a disjoint collection. So $\{I_k\}_{k=1}^\infty$ is a countable disjoint subcollection of $\mathcal{F}$ such that for each $n \in \mathbb{N}$,

$$\ell(I_{n+1}) > \ell(I)/2 \text{ if } I \in \mathcal{F}_n \text{ and } I \cap (\cup_{k=1}^n I_K) = \varnothing. \qquad (6)$$

The Vitali Covering Lemma (continued 3)

Proof (continued). We infer from (3) that $\{\ell(I_k)\} \to 0$. Fix $n \in \mathbb{N}$.

To verify the inclusion (5), let $x \in E \setminus (\cup_{k=1}^{n} I_k)$. We infer from (4) that there is $I \in \mathcal{F}$ which contains x and is disjoint from $\cup_{k=1}^{n} I_k$ (in fact, $I \in \mathcal{F}_n$). Now I must have nonempty intersection with some I_k (where $k > n$), for otherwise $I \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and by (6), $\ell(I_k) > \ell(i)/$ for all $k \in \mathbb{N}$, contradicting the convergence of $\{\ell(I_k)\}$ to 0. Let N be the first natural number for which $I \cap I_N \neq \varnothing$. Then $> n$. Since $I \cap (\cup_{k=1}^{N-1} I_k) = \varnothing$, we infer from (6) that $\ell(I_N) > \ell(I)/2$. Since $x \in I$ and $I \cap I_N \neq \varnothing$, the distance from x to the midpoint of I_N is at most $\ell(I) + \ell(I_N)/2$:



The Vitali Covering Lemma (continued 4)

Proof (continued). Hence, since $\ell(I) < 2\ell(I_N)$, the distance from x to the midpoint of I_N is less than $(5/2)\ell(I_N)$. This means that x belongs to $5 * I_N$. Thus, since $N \geq n+1$ then $x \in 5 * I_N \subseteq \cup_{k=N+1}^{\infty} 5 * I_k$ and we have established (5).

Let $\varepsilon > 0$. We infer from (3) that there is $n \in \mathbb{N}$ for which $\sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/5$. For this n, we have by (5) that $E \setminus (\cup_{k=1}^{n} I_k) \subseteq \sup_{k=n+1}^{\infty} 5 * I_k$ and by monotonicity of outer measure

$$\begin{aligned} m^*\left(E \setminus (\cup_{k=1}^{n} I)K)\right) &\leq m^*\left(\cup_{k=n+1}^{\infty} 5 * I_k\right) \\ &\leq \sum_{k=n+1}^{\infty} 5\ell(I_k) \text{ by subadditivity} \\ &= 4(\varepsilon/5) = \varepsilon. \end{aligned}$$

This proves (2). □

Lemma 6.3

Lemma 6.3. Let f be an increasing function on the closed, bounded interval $[a, b]$. Then for each $\alpha > 0$,

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)] \tag{7}$$

and

$$m^*\{x \in (a, b) \mid \overline{D}[f(x)] = \infty\} = 0. \tag{8}$$

Proof. Let $\alpha > 0$. Define $E_\alpha = \{x \in (a, b) \mid \overline{D}[f(x)] \geq \alpha\}$. Choose $\alpha' \in (0, \alpha)$. Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c, d]$ contained in (a, b) for which $f(d) - f(c) \geq \alpha'(d - c)$. Since $\overline{D}[f(x)] \geq \alpha$ then we have by definition, that for each $x \in E_\alpha$ there is $\delta > 0$ such that $\sup_{0<|t|\leq\delta}(f(x+t) - f(x))/t \geq \alpha - \alpha' = \varepsilon$ (where we take $\varepsilon = \alpha - \alpha' > 0$), which implies $f(x+\delta) - f(x) \geq \delta\alpha$ (with $t = \delta$) and $f(x) - f(x-\delta) \geq \delta\alpha$ (with $t = -\delta$).

Lemma 6.3 (continued 1)

Proof (continued). So with $d = x + \delta$ and $c = x - \delta$ ($\delta > 0$) these inequalities yield $f(d) - f(x) \geq \alpha(d - x)$ and $f(x) - f(c) \geq \alpha(x - c)$. Adding we have $f(d) - f(c) \geq \alpha(d - c)$. So $[c, d] \in \mathcal{F}$ (where we choose δ sufficiently small so that $[c, d] \subset (a, b)$. Now the above argument holds for any $\delta' < \delta$, $\delta' > 0$, and so interval $[c, d]$ containing $x \in E_\alpha$ can be made arbitrarily small in length. So $\mathcal{F}$ is a covering of E_α in the sense of Vitali.

The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of $\mathcal{F}$ for which $m^*[E_\alpha \setminus (\cup_{k=1}^{n}[c_k, d_k]] < \varepsilon$ for any given $\varepsilon > 0$. Since

$$E_\alpha \subseteq (\cup_{k=1}^{n}[c_k, d_k]) \cup \left(E_\alpha \setminus \left(\sup_{k=1}^{n}[c_k, d_k[\right)\right)$$

then by finite subadditivity

$$m^*(E_\alpha) \leq \sum_{k=1}^{n}(d_k - c_k) + \varepsilon \leq \frac{1}{\alpha'}\sum_{k=1}^{n}[f(d_k) - f(c_k)] + \varepsilon.$$

Lemma 6.3 (continued 2)

Proof (continued). However, the function f is increasing on $[a,b]$ and $\{[c_k,d_k]\}_{k=1}^n$ is a disjoint collection of subintervals of $[a,b]$, so

$$\begin{aligned}\sum_{k=1}^n [f(d_k)-f(c_k)] &= (f(d_1)-f(c_1))+(f(d_2)-f(c_2))+\cdots \\ &\quad +(f(d_{n-1})-f(c_{c-1})+(f(d_n)-f(c_n)) \\ &= -f(c_1)+(f(d_1)-f(c_2)+\cdots+(f(d_{n-1})-f(c_n))+f(d_n) \\ &\leq -f(c_1)+0+0+\cdots+0+f(d_n) \text{ (where we index} \\ &\text{the } c_k, d_k\text{'s in increasing order; because } f \text{ is an increasing function)} \\ &\leq f(b)-f(a) \text{ since } a<c_1,\ d_n<b \text{ and } f \text{ is increasing.}\end{aligned}$$

Thus, for each $\varepsilon>0$ and $\alpha'\in(0,\alpha)$ we have $m^*(E_\alpha)\leq(1/\alpha')[f(b)-f(a)]+\varepsilon$, and so (7) holds.

Lemma 6.3 (continued 3)

Lemma 6.3. Let f be an increasing function on the closed, bounded interval $[a,b]$. Then for each $\alpha>0$,

$$m^*\{x\in(a,b)\mid \overline{D}[f(x)]\geq\alpha\}\leq\frac{1}{\alpha}[f(b)-f(a)] \qquad (7)$$

and

$$m^*\{x\in(a,b)\mid \overline{D}[f(x)]=\infty\}=0. \qquad (8)$$

Proof (continued). For all $n\in\mathbb{N}$ we have that $\{x\in(a,b)\mid\overline{D}[f(x)]=\infty\}\subseteq E_n$ and therefore by monotonicity of m^* and (7) we have

$$m^*(\{x\in(a,b)\mid\overline{D}[f(x)]=\infty\})\leq m^*(E_n)\leq\frac{1}{n}[f(b)-f(a)],$$

and so $m^*(\{x\in(a,b)\mid\overline{D}[f(x)]=\infty\})=0$ and (8) holds. □

Lebesgue Theorem

Lebesgue Theorem.
If the function f is monotone on the open interval (a,b), then it is differentiable almost everywhere on (a,b).

Proof. Let f be increasing on (a,b). WLOG, we take f to be bounded (otherwise we express (a,b) as the union of an ascending sequence of open, bounded intervals on which f is increasing and bounded, show that f is differentiable a.e. on each of these intervals, and then apply continuity of Lebesgue measure (theorem 2.15) to the sequence of measure zero subsets of the ascending sequence of intervals and again conclude that f is a.e. differentiable). The set of points $x\in(a,b)$ at which $\overline{D}[f(x)]>\underline{D}[f(x)]$ is the union of the sets $E_{\alpha,\beta}=\{x\in(a,b)\mid\overline{D}[f(x)]>\alpha>\beta>\underline{D}[f(x)]\}$ where $\alpha,\beta\in\mathbb{Q}$.

Lebesgue Theorem (continued 1)

Proof (continued). Fix $\alpha,\beta\in\mathbb{Q}$ with $\alpha>\beta$ and set $E=E_{\alpha,\beta}$. Let $\varepsilon>0$. Choose an open set $\mathcal{O}$ for which

$$E\subseteq\mathcal{O}\subseteq(a,b) \text{ and } m(\mathcal{O})<m^*(E)+\varepsilon \qquad (10)$$

(which can be done by the infimum definition of outer measure). Let $\mathcal{F}$ be the collection of closed, bounded intervals $[c,d]$ contained in $\mathcal{O}$ for which $f(d)-f(c)<\beta(d-c)$. Since $\underline{D}[f(x)]<\beta$ on E, then as explained in the proof of Lemma 6.3, $\mathcal{F}$ is a Vitali covering of E. The Vitali Covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k,d_k]\}_{k=1}^n$ of $\mathcal{F}$ for which $m^*[E\setminus(\cup_{k=1}^n[c_k,d_k])]<\varepsilon$. By the choice of the intervals $[c_k,d_k]$, we have

$$\begin{aligned}\sum_{k=1}^n[f(d_k)-f(c_k)] &< \beta\sum_{k=1}^n(d_j-c_k) \\ &\leq \beta m(\mathcal{O}) \text{ by monotonicity of } m \\ &\leq \beta[m^*(E)+\varepsilon] \text{ by (10).} \qquad (12)\end{aligned}$$

Lebesgue Theorem (continued 2)

Proof (continued). For $1 \le k \le n$, we have by Lemma 6.3 applied to the restriction of f to $[c_k, d_k]$, that

$$m^*(E \cap (c_k, d_k)) \le (1/\alpha)[f(d_k) - f(c_k)] \qquad (12')$$

since $\overline{D}[f(x)] > \alpha$ on (c_k, d_k). Therefore, by (11) and subadditivity

$$\begin{aligned} m^*(E) &= m^*\left[(E \cap (\cup_{k=1}^n (c_k, d_k))) \cup (E \setminus \cup_{k=1}^n [c_k, d_k])\right] \\ &\le \sum_{k=1}^n m^*(E \cap (c_k, d_k)) + \varepsilon \\ &\le \frac{1}{\alpha}\sum_{k=1}^n [f(d_k) - f(c_k)] + \varepsilon \text{ by } (12'). \qquad (13) \end{aligned}$$

Lebesgue Theorem (continued 3)

Lebesgue Theorem.
If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

Proof (continued). We infer from (12) and (13) that

$$m^*(E) \le \frac{1}{\alpha}[\beta(m^*(E) + \varepsilon)] + \varepsilon = \frac{\beta}{\alpha} m^*(E) + \frac{\beta}{\alpha}\varepsilon + \varepsilon$$

for all $\varepsilon > 0$. Therefore, since $0 \le m^*(E), \infty$ and $\beta/\alpha < 1$, then it must be that $m^*(E) = 0$. The general result now holds as described at the beginning of the proof. □

Corollary 6.4

Corollary 6.4. Let f be an increasing function on the closed, bounded interval $[a, b]$. Then f' is integrable over $[a, b]$ and

$$\int_a^b f' \le f(b) - f(a).$$

Proof. Since f is increasing on $[a, b+1]$ (after extending it) then f is measurable by Problem 6.22 and therefore $\mathrm{Diff}_h[f(x)]$ is measurable (it is a linear combination of measurable functions). Lebesgue's Theorem tells us that f is differentiable a.e. on (a, b). Therefore $\{\mathrm{Diff}_{1/n}[f]\}_{n=1}^\infty$ is a sequence of nonnegative measurable functions that converges pointwise a.e. on $[a, b]$ to f'. By Fatou's Lemma

$$\int_a^b f' \le \liminf_{n\to\infty} \left(\int_a^b \mathrm{Diff}_{1/n}[f]\right). \qquad (16)$$

Corollary 6.4 (continued 1)

Corollary 6.4. Let f be an increasing function on the closed, bounded interval $[a, b]$. Then f' is integrable over $[a, b]$ and

$$\int_a^b f' \le f(b) - f(a).$$

Proof (continued). Notice that for $a \le u < v \le b$ we have by a change of variables that

$$\begin{aligned} \int_u^v \mathrm{Diff}_h[f] &= \int_u^v \frac{f(x+h) - f(x)}{h} = \frac{1}{h}\int_u^v (f(x+h) - f(x)) \\ &= \frac{1}{h}\left(\int_{u+h}^v f(x) + \int_v^{v+h} f(x) - \int_u^{u+h} f(x) - \int_{u+h}^v f(x)\right) \\ &\quad \text{(notice that } f \text{ is increasing on } [a,b], \\ &\quad \text{ so } f \text{ is integrable on } [a,b] \text{ and linearity holds)} \\ &= \frac{1}{h}\left(\int_v^{v+h} f(x) - \int_u^{u+h} f(x)\right) = \frac{1}{h}(\mathrm{Av}_h f(v) - \mathrm{Av}_h f(u)). \end{aligned}$$

Corollary 6.4 (continued 2)

Proof (continued). So

$$\begin{aligned}\mathrm{Diff}_{1/n}[f] &= \frac{1}{1/n}\int_b^{b+1/n} f(x) - \frac{1}{1/n}\int_a^{a+1/n} f(x) \\ &= f(b) - \frac{1}{1/n}\int_a^{a+1/n} f(x) \text{ since } f(x) \equiv f(b) \text{ on } [b, b+1/n] \\ &\le f(b) - f(a) \text{ since } f \text{ is increasing on } [a, b]\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\left(\int_a^b \mathrm{Diff}_{1/n}[f]\right) \le f(b) - f(a). \qquad (17)$$

Combining (16) and (17) gives

$$\int_a^b f' \le \liminf_{n\to\infty}\left(\int_a^b \mathrm{Diff}_{1/n}[f]\right)$$

$$\le \limsup_{n\to\infty}\left(\int_a^b \mathrm{Diff}_{1/n}[f]\right) \le f(b) - f(a). \quad \square$$