Lemma 6.5

The function \( f(x) \) is bounded variation on \([a, b]\) if and only if there exist \( n \) partitions \( P \) and \( Q \) of \([a, b]\) such that:

\[
\sum_{k=1}^{n} |f(x_k^+) - f(x_k^-)| = \sup_{P \in \mathcal{P}[a,b]} \left\{ \sum_{k=1}^{n} |f(x_k^+) - f(x_k^-)| \right\} = \inf_{Q \in \mathcal{Q}[a,b]} \left\{ \sum_{k=1}^{n} |f(x_k^+) - f(x_k^-)| \right\}
\]

where \( \mathcal{P}[a,b] \) and \( \mathcal{Q}[a,b] \) denote the sets of partitions of \([a, b]\).
By Corollary 6.4, if \( f \) and \( g \) is integrable on \([a, b]\) and \( f \geq g \), then \( f \) is differentiable on \([a, b]\) and hence \( f \) is differentiable on \([a, b]\). By Lebesgue's Theorem, each increasing function is integrable on \([a, b]\). By Jordan's Theorem, \( f \) is the difference of two increasing functions on \([a, b]\).

Corollary 6.6. If the function \( f \) is of bounded variation on the closed interval \([a, b]\) and \( y \) is a bounded variation over \([a, b]\), then it is differentiable almost everywhere on the bounded interval \([a, b]\).

\[
\begin{align*}
|((1-t)x + (1-t)y) - (tx) - (tx)| & \geq \left( \int_{a}^{b} \right) = \\
|((1-t)x) - (tx)| & \geq (d, f) \Lambda
\end{align*}
\]

Proof (continued).

For any partition \( \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\) where \( q \in (a, b) \) and \( y \) is integrable functions on \([a, b]\), the difference of two increasing functions. For the converse, let \( f \) be a Jordan decomposition of \( f \). Then, by Lemma 6.5, the function \( f \) is of bounded variation on the closed interval \([a, b]\).