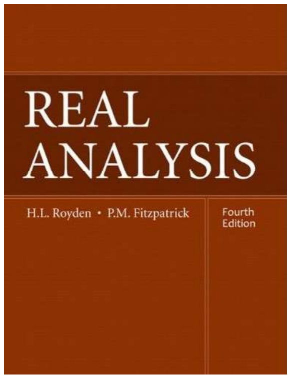
Real Analysis

Chapter 6. Differentiation and Integration

6.3. Functions of Bounded Variation: Jordan's Theorem—Proofs of Theorems



Lemma 6.5

Lemma 6.5. Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a, b]$:

$$f(x) = [f(x) + TV(f_{[a,x]})] - TV(f_{[a,x]}) \text{ for all } x \in [a, b].$$

Proof. Notice that if $c \in (a, b)$, P is a partition of $[a, b]$, and P' is the refinement of P obtained by adjoining c to P, then by the Triangle Inequality,

$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1, i \neq j}^{k} |f(x_i) - f_{i-1}| + |f(x_j) - f(x_{j-1})| \end{aligned}$$

Lemma 6.5 (continued 1)

Proof (continued).

$$\begin{aligned} V(f, P) &= \sum_{i=1, i \neq j}^{k} |f(x_i) - f_{i-1})| + |f(x_j) - f(c) + f(c) - f(x_{j-1})| \\ &\qquad \text{where } x_{j-1} < c < x_j \\ &\le \sum_{i=1, i \neq j}^{k} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{j-1}| + |f(x_j) - f(c)| \\ &= V(f, P') \end{aligned}$$

Since $TV(f)$ is defined in terms of suprema over all partitions of $[a, b]$ and since any partition P refined by adding c yields $V(f, P) \le V(f, P')$, then $TV(f)$ can be computed by taking suprema over partitions containing point c. Now a partition P of $[a, b]$ that contains the point c induces, and is induced by, partitions P_1 and P_2 of $[a, c]$ and $[c, b]$, respectively.

Lemma 6.5 (continued 2)

Proof (continued). For such partitions, $V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2)$ (where $V(f_{[a,c]}, P_1)$ denotes the variation of the restriction of f to $[a, c]$ with respect to partition P_1 of $[a, c]$). Take the supremum among such partitions P_1 and P_2 to conclude that $TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]})$. Since this holds for any $c \in (a, b)$, we have for all $a \le u < v \le b$ that $TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]})$ and since f is of bounded variation,

$$TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]}) \ge 0 \text{ for all } a \le u < v \le b. \qquad (21)$$

So the function $x \mapsto TV(f_{[z,x]})$ is a real-valued increasing function (since (21) shows that this function evaluated at v is greater than or equal to this function evaluated at u).

Lemma 6.5 (continued 3)

Lemma 6.5. Let the function f be of bounded variation on the closed, bounded interval $[a, b]$. Then f has the following explicit expression as the difference of two increasing functions on $[a, b]$:

$$f(x) = [f(x) + TV(f_{[a,x]})] - TV(f_{[a,x]}) \text{ for all } x \in [a, b].$$

Proof (continued). By considering the partition $P = \{u, v\}$ of $[u, v]$ we have

$$\begin{aligned} f(u) - f(v) &\leq |f(v) - f(u)| = V(f_{[u,v]}, P) \leq TV(f_{[u,v]}) \\ &= TV(f_{[a,v]}) - TV(f_{[a,u]}) \text{ by (21).} \end{aligned}$$

Thus $f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]})$ for all $a \leq u < v \leq b$. So the function $x \mapsto f(x) + TV(f_{[a,x]})$ is an increasing function. So we can write $f(x)$ as $f(x) = [f(x) + TV(f_{[a,x]})] - [TV(f_{[a,x]})]$ for $x \in [a, b]$ where the function sin square brackets are both increasing. □

Jordan's Theorem

Jordan's Theorem.
A function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$. When f is written as such a difference, it is called a *Jordan decomposition* of f.

Proof. Let f be of bounded variation on $[a, b]$. Lemma 6.5 shows that f is the difference of two increasing functions. For the converse, let $f = g - h$ on $[a, b]$ where g and h are increasing functions on $[a, b]$. For any partition $P = \{x_0, x_1, \ldots, x_k\}$ of $[a, b]$,

$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{k} |g(x_i) - h(x_i) - g(x_{i-1}) + h(x_{i-1})| \end{aligned}$$

Jordan's Theorem

Proof (continued).

$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} [(g(x_i) - g(x_{i-1})) + (h(x_{i-1}) - h(x_i))] \\ &= \sum_{i=1}^{k} |g(x_i) - g(x_{i-1}| + \sum_{i=1}^{k} |h(x_i - h(x_{i-1})| \\ &= \sum_{i=1}^{k} (g(x_i) - g(x_{i-1}) + \sum_{i=1}^{k} (h(x_i - h(x_{i-1})) \\ &\qquad \text{since } g \text{ and } h \text{ are increasing} \\ &= (g(b) - g(a)) + (h(b) - h(a)). \end{aligned}$$

So $TV(f) \leq (g(b) - g(a)) + (h(b) - h(a))$ and f is of bounded variation on $[a, b]$. □

Corollary 6.6

Corollary 6.6. If the function f is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval (a, b) and f' is integrable over $[a, b]$.

Proof. By Jordan's Theorem, f is the difference of two increasing functions on $[a, b]$. By Lebesgue's Theorem, each increasing function is differentiable a.e. on (a, b) and hence f itself is a.e. differentiable on (a, b). By Corollary 6.4, $\int_{[a,b]} f' \leq f(b) - f(a)$ and f' is integrable on $[a, b]$. □