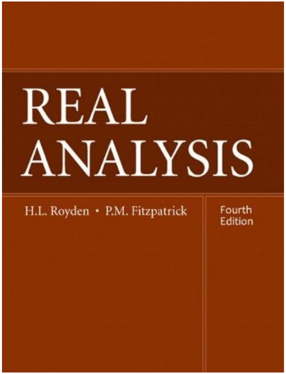
Real Analysis

Chapter 6. Differentiation and Integration

6.3. Functions of Bounded Variation: Jordan's Theorem—Proofs of Theorems



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$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1, i\neq j}^{k} |f(x_i) - f_{i-1}| + |f(x_j) - f(x_{j-1})| \end{aligned}$$

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$$\begin{aligned} V(f,P) &= \sum_{i=1, i\neq j}^{k} |f(x_i) - f_{i-1})| + |f(x_j) - f(c) + f(c) - f(x_{j-1})| \\ &\qquad\qquad\qquad \text{where } x_{j-1} < c < x_j \\ &\leq \sum_{i=1, i\neq j}^{k} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{j-1}| + |f(x_j) - f(c)| \\ &= V(f,P') \end{aligned}$$

Since $TV(f)$ is defined in terms of suprema over all partitions of $[a,b]$ and since any partition P refined by adding c yields $V(f,P) \leq V(f,P')$, then $TV(f)$ can be computed by taking suprema over partitions containing point c. Now a partition P of $[a,b]$ that contains the point c induces, and is induced by, partitions P_1 and P_2 of $[a,c]$ and $[c,b]$, respectively.

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Proof (continued). For such partitions, $V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2)$ (where $V(f_{[a,c]}, P_1)$ denotes the variation of the restriction of f to $[a, c]$ with respect to partition P_1 of $[a, c]$). Take the supremum among such partitions P_1 and P_2 to conclude that $TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]})$. Since this holds for any $c \in (a, b)$, we have for all $a \leq u < v \leq b$ that $TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]})$ and since f is of bounded variation,

$$TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]}) \geq 0 \text{ for all } a \leq u < v \leq b. \qquad (21)$$

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$$\begin{aligned} f(u) - f(v) &\leq |f(v) - f(u)| = V(f_{[u,v]}, P) \leq TV(f_{[u,v]}) \\ &= TV(f_{[a,v]}) - TV(f_{[a,u]}) \text{ by (21).} \end{aligned}$$

Thus $f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]})$ for all $a \leq u < v \leq b$. So the function $x \mapsto f(x) + TV(f_{[a,x]})$ is an increasing function. So we can write $f(x)$ as $f(x) = [f(x) + TV(f_{[a,x]})] - [TV(f_{[a,x]})]$ for $x \in [a, b]$ where the function sin square brackets are both increasing. □

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A function f is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$. When f is written as such a difference, it is called a *Jordan decomposition* of f.

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$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{k} |g(x_i) - h(x_i) - g(x_{i-1}) + h(x_{i-1})| \end{aligned}$$

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V(f,P) &= \sum_{i=1}^{k}[(g(x_i)-g(x_{i-1}))+(h(x_{i-1})-h(x_i))] \\
&= \sum_{i=1}^{k}|g(x_i)-g(x_{i-1}|+\sum_{i=1}^{k}|h(x_i-h(x_{i-1})| \\
&= \sum_{i=1}^{k}(g(x_i)-g(x_{i-1})+\sum_{i=1}^{k}(h(x_i-h(x_{i-1})) \\
&\qquad\qquad \text{since } g \text{ and } h \text{ are increasing} \\
&= (g(b)-g(a))+(h(b)-h(a)).
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So $TV(f) \le (g(b)-g(a))+(h(b)-h(a))$ and f is of bounded variation on $[a,b]$. □

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Proof. By Jordan's Theorem, f is the difference of two increasing functions on $[a, b]$. By Lebesgue's Theorem, each increasing function is differentiable a.e. on (a, b) and hence f itself is a.e. differentiable on (a, b).

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