Chapter 6. Differentiation and Integration
6.3. Functions of Bounded Variation: Jordan’s Theorem—Proofs of Theorems
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Lemma 6.5

Lemma 6.5. Let the function $f$ be of bounded variation on the closed, bounded interval $[a, b]$. Then $f$ has the following explicit expression as the difference of two increasing functions on $[a, b]$:

$$f(x) = [f(x) + TV(f_{[a,x]})] - TV(f_{[a,x]}) \text{ for all } x \in [a, b].$$

Proof. Notice that if $c \in (a, b)$, $P$ is a partition of $[a, b]$, and $P'$ is the refinement of $P$ obtained by adjoining $c$ to $P$, then by the Triangle Inequality,
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$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{k} |f(x_i) - f_{i-1}| + |f(x_j) - f(x_{j-1})|$$
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Lemma 6.5 (continued 1)

Proof (continued).

\[ V(f, P) = \sum_{i=1, i \neq j}^{k} |f(x_i) - f_{i-1}| + |f(x_j) - f(c) + f(c) - f(x_{j-1})| \]

where \( x_{j-1} < c < x_j \)

\[ \leq \sum_{i=1, i \neq j}^{k} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{j-1})| + |f(x_j) - f(c)| \]

\[ = V(f, P') \]

Since \( TV(f) \) is defined in terms of suprema over all partitions of \([a, b]\) and since any partition \( P \) refined by adding \( c \) yields \( V(f, P) \leq V(f, P') \), then \( TV(f) \) can be computed by taking suprema over partitions containing point \( c \). Now a partition \( P \) of \([a, b]\) that contains the point \( c \) induces, and is induced by, partitions \( P_1 \) and \( P_2 \) of \([a, c]\) and \([c, b]\), respectively.
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Since \( TV(f) \) is defined in terms of suprema over all partitions of \([a, b]\) and since any partition \( P \) refined by adding \( c \) yields \( V(f, P) \leq V(f, P') \), then \( TV(f) \) can be computed by taking suprema over partitions containing point \( c \). Now a partition \( P \) of \([a, b]\) that contains the point \( c \) induces, and is induced by, partitions \( P_1 \) and \( P_2 \) of \([a, c]\) and \([c, b]\), respectively.
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**Proof (continued).** For such partitions,
\[ V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2) \]
(where \( V(f_{[a,c]}, P_1) \) denotes the variation of the restriction of \( f \) to \([a, c] \) with respect to partition \( P_1 \) of \([a, c] \)). Take the supremum among such partitions \( P_1 \) and \( P_2 \) to conclude that \( TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]}) \). Since this holds for any \( c \in (a, b) \), we have for all \( a \leq u < v \leq b \) that
\[ TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]}) \]
and since \( f \) is of bounded variation,
\[ TV(f_{[a,v]}) = TV(f_{[a,u]}) + TV(f_{[u,v]}) \geq 0 \text{ for all } a \leq u < v \leq b. \]  
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So the function \( x \mapsto TV(f_{[z,x]}) \) is a real-valued increasing function (since (21) shows that this function evaluated at \( v \) is greater than or equal to this function evaluated at \( u \)).
Proof (continued). For such partitions, 
\[ V(f_{[a,b]}, P) = V(f_{[a,c]}, P_1) + V(f_{[c,b]}, P_2) \] (where \( V(f_{[a,c]}, P_1) \) denotes the variation of the restriction of \( f \) to \([a, c]\) with respect to partition \( P_1 \) of \([a, c]\)). Take the supremum among such partitions \( P_1 \) and \( P_2 \) to conclude that \( TV(f_{[a,b]}) = TV(f_{[a,c]}) + TV(f_{[c,b]}) \). Since this holds for any \( c \in (a, b) \), we have for all \( a \leq u < v \leq b \) that 
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Lemma 6.5. Let the function $f$ be of bounded variation on the closed, bounded interval $[a, b]$. Then $f$ has the following explicit expression as the difference of two increasing functions on $[a, b]$:

$$f(x) = [f(x) + TV(f_{[a,x]}))] - TV(f_{[a,x]})$$

for all $x \in [a, b]$.

Proof (continued). By considering the partition $P = \{u, v\}$ of $[u, v]$ we have

$$f(u) - f(v) \leq |f(v) - f(u)| = V(f_{[u,v]}, P) \leq TV(f_{[u,v]})$$

$$= TV(f_{[a,v]}) - TV(f_{[a,u]})$$

by (21).

Thus $f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]})$ for all $a \leq u < v \leq b$. So the function $x \mapsto f(x) + TV(f_{[a,x]})$ is an increasing function. So we can write $f(x)$ as $f(x) = [f(x) + TV(f_{[a,x]}))] - [TV(f_{[a,x]}))]$ for $x \in [a, b]$ where the function sin square brackets are both increasing.
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Thus $f(v) + TV(f_{[a,v]}) \geq f(u) + TV(f_{[a,u]})$ for all $a \leq u < v \leq b$. So the function $x \mapsto f(x) + TV(f_{[a,x]})$ is an increasing function. So we can write $f(x)$ as $f(x) = [f(x) + TV(f_{[a,x]}))] - [TV(f_{[a,x]})]$ for $x \in [a, b]$ where the function sin square brackets are both increasing. \qed
Jordan’s Theorem

A function $f$ is of bounded variation on the closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$. When $f$ is written as such a difference, it is called a Jordan decomposition of $f$.

Proof. Let $f$ be of bounded variation on $[a, b]$. Lemma 6.5 shows that $f$ is the difference of two increasing functions. For the converse, let $f = g - h$ on $[a, b]$ where $g$ and $h$ are increasing functions on $[a, b]$. 
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V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|
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Proof (continued).

\[ V(f, P) = \sum_{i=1}^{k} [(g(x_i) - g(x_{i-1})) + (h(x_{i-1}) - h(x_i))] \]

\[ = \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| \]

\[ = \sum_{i=1}^{k} (g(x_i) - g(x_{i-1})) + \sum_{i=1}^{k} (h(x_i) - h(x_{i-1})) \]

since \( g \) and \( h \) are increasing

\[ = (g(b) - g(a)) + (h(b) - h(a)). \]

So \( TV(f) \leq (g(b) - g(a)) + (h(b) - h(a)) \) and \( f \) is of bounded variation on \([a, b]\).
Proof (continued).

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Corollary 6.6. If the function $f$ is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval $(a, b)$ and $f'$ is integrable over $[a, b]$.

**Proof.** By Jordan’s Theorem, $f$ is the difference of two increasing functions on $[a, b]$. By Lebesgue’s Theorem, each increasing function is differentiable a.e. on $(a, b)$ and hence $f$ itself is a.e. differentiable on $(a, b)$.  


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