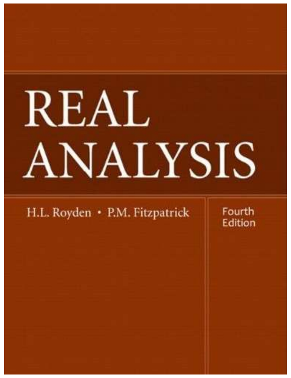
Real Analysis

Chapter 6. Differentiation and Integration

6.4. Absolutely Continuous Functions—Proofs of Theorems



Proposition 6.7

Proposition 6.7. If the function f is Lipschitz on a closed, bounded interval $[a,b]$, then it is absolutely continuous on $[a,b]$.

Proof. Let $c > 0$ be a Lipschitz constant for f on $[a,b]$. That is, $f(u) - f(v)| \le c|u - v|$ for all $u, v \in [a,b]$. Then with $\delta = \varepsilon/c$, the definition of absolute continuity is satisfied. □

Theorem 6.8

Theorem 6.8. Let the function f be absolutely continuous on the closed, bounded interval $[a,b]$. Then f is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

Proof. For bounded variation, let δ correspond to $\varepsilon = 1$ in the definition of absolute continuity of f. Let P be a partition of $[a,b]$ into N closed intervals $\{[c_k, d_k]\}_{k=1}^N$, each of length less than δ. Since we chose $\varepsilon = 1$, we have $TV(f_{[c_k,d_k]}) \le 1$ for $1 \le k \le n$. The additivity formula (19) in the proof of Lemma 6.5 then gives $TV(f) = \sum_{k=1}^N TV(f_{[c_k,d_k]}) \le N$. Therefore, f is of bounded variation on $[a,b]$.

Now for the "difference of increasing absolutely continuous functions" part. Since f is of bounded variation, by Lemma 6.5, $f(x) = [f(x) + TV(f_{[a,x]})] - [TV(f_{[a,x]})]$. So the claim follows if $TV(f_{[a,x]})$ is absolutely continuous.

Theorem 6.8 (continued)

Proof (continued). Let $\varepsilon > 0$. Choose δ to correspond to $\varepsilon/2$ in the definition of absolute continuity of f on$[a,b]$. Let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of (a,b) for which $\sum_{k=1}^n (d_k - c_k) < \delta$. For $1 \le k \le n$, let P_k be a partition of $[c_k, d_k]$. Then $\sum_{k=1}^n V(f_{[c_k,d_k]}, P_k) < \varepsilon/2$ because $V(f_{[c_k,d_k]}, P_k)$ involves a portion of the differences in function values in the definition of absolute continuity and the sum of the differences of the x values in P_k is $d_k - c_k$, so the the choice of δ we have this sum of differences of function values less than $\varepsilon/2$. Letting P_k vary over all partitions of $[c_k, d_k]$ gives that $\sum_{k=1}^n TV(f_{[c_k,d_k]}) \le \varepsilon/2 < \varepsilon$. We infer from (21) in the proof of Lemma 6.5 (see page 117) that for $1 \le k \le n$, $TV(f_{[c_k,d_k]}) = TV(f_{[a,d_k]}) - TV(f_{[a,c_k]})$. Hence, if $\sum_{i=1}^n (d_k - c_k) < \delta$ then $\sum_{k=1}^n |TV(f_{[a,d_k]}) - TV(f_{[a,c_k]})| = \sum_{k=1}^n TV(f_{[c_k,d_k]}) \le \varepsilon/2 < \varepsilon$. So the total variation function $TV(f_{[a,x]})$ is absolutely continuous. □

Theorem 6.9

Theorem 6.9. Let the function f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$.

Proof. First, assume $\{\text{Diff}_h[f]\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$. Let $\varepsilon > 0$. Choose $\delta > 0$ for which (by the definition of uniform integrability) $\in_E |\text{Diff}_h[f]| < \varepsilon/2$ if $m(E) < \delta$ and $0 < k \le 1$. To show absolute continuity, let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of (a, b) for which $\sum_{k=1}^n (d_k - c_k) < \delta$. For $0 < h \le 1$ and $1 \le k \le n$ we have by the change of variables formula (14) (also in the proof of Corollary 6.4), $\text{Av}_h[f(d_k)] - \text{Av}_h[f(c_k)] = \int_{c_k}^{d_k} \text{Diff}_h[f]$. Therefore $\sum_{k=1}^n |\text{Av}_h[f(d_k)] - \text{Av}_h[f(c_k)]| \le \sum_{k=1}^n \int_{c_k}^{d_k} |\text{Diff}_h[f]| = \int_E |\text{Diff}_h[f]|$ (by additivity) where $E \cup_{k=1}^n (c_k, d_k)$ has measure less than δ.

Theorem 6.9 (continued 1)

Proof (continued). So by the choice of δ,

$$\sum_{k=1}^n |\text{Av}_h[f(d_k)] = \text{Av}_h[f(c_k)]| = \int_E |\text{Diff}_h[f]| < \varepsilon/2 \quad (*)$$

for all $0 < h \le 1$. Since f is continuous then $\lim_{h\to 0^+} \text{Av}_h[f] = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f = f(x)$ (as in Calculus 1), so taking a limit as $h \to 0^+$ in $(*)$ we get $\sum_{k=1}^n |f(d_k) - f(c_k)| < \varepsilon/2 < \varepsilon$. Hence, f is absolutely continuous on $[a, b]$.

Now suppose f is absolutely continuous. By Theorem 6.8, f is the difference of two increasing absolutely continuous functions, say $f = f_1 - f_2$. So we can assume WLOG that f is increasing (because $\text{Diff}_h[f] = \text{Diff}_h[f_1] - \text{Diff}_h[f_2]$ and if families $\{\text{Diff}_h[f_1]\}_{0<h\le 1}$ and $\{\text{Diff}_h[f_2]\}_{0<h\le 1}$ are uniformly integrable, then $\{\text{Diff}_h[f_1 - f_2]\}_{0<h\le 1}$ is uniformly integrable [let $\varepsilon > 0$ and choose δ_1 and δ_2 for f_1 and f_2 to "correspond" to $\varepsilon/2$, then choose $\delta = \min\{\delta_1, \delta_2\}$]).

Theorem 6.9 (continued 2)

Proof (continued). Therefore the divided differences (WLOG) are nonnegative: $\text{Diff}_h[f] = (f(x+h) - f(x))/h \ge 0$. Let $\varepsilon > 0$. To show uniform integrability, we must find $\delta > 0$ such that for each measurable subset E of (a, b),

$$\int_E \text{Diff}_h[f] < \varepsilon \text{ if } m(E) < \varepsilon \text{ and } 0 < h \le 1. \quad (25)$$

By Theorem 2.11, measurable set E is contained in a G_δ set G for which $m(G \setminus E) = 0$. But every G_δ set is the intersection of a descending sequence of open sets ($G = \cap_{k=1}^\infty \mathcal{O}_k$, so take the sequence of open sets as $O_n = \cap_{k=1}^m \mathcal{O}_k$). Moreover, every open set is the disjoint union of a countable collection of open intervals, and therefore every open set is the union of an ascending sequence of open sets, each of which is the union of a *finite* disjoint collection of open intervals (if the open set is $\cup_{k=1}^\infty I_k$ where I_k are open intervals, then define $O_n = \cup_{k=1}^n I_k$ and the open set is $\lim_{n\to\infty} O_n$ and $\{O_n\}_{n=1}^\infty$ is an ascending sequence).

Theorem 6.9 (continued 3)

Proof (continued). Below we will show that there is $\delta > 0$ such that for $\{(c_k, d_k)\}_{k=1}^n$ a finite disjoint collection of open subintervals of (a, b):

$$\int_E \text{Diff}_h[f] < \varepsilon/2 \text{ if } m(E) < \delta \text{ where } E = \cup_{k=1}^n (c_k, d_k) \text{ and } 0 < h \le 1. \quad (26)$$

If (26) is established, then for E an open set with $E = \cup_{k=1}^\infty (c_k, d_k)$ we have by continuity of the integral (Theorem 4.21) that is $m(E) < \delta$ and $0 < h \le 1$ then

$$\int_E \text{Diff}_h[f] = \int_{\cup_{k=1}^\infty (c_k, d_k)} \text{Diff}_h[f] = \lim_{n\to\infty} \left(\int_{\cup_{k=1}^n (c_k, d_k)} \text{Diff}_n[f] \right) \le \frac{\varepsilon}{2}. \quad (26')$$

Now with (26′) established, if $G = \cap_{k=1}^\infty O_k$ is a G_δ set where $m(G \setminus E) = 0$, then...

Theorem 6.9 (continued 4)

Proof (continued).

$$\begin{aligned}\int_E \text{Diff}_h[f] &\le \int_G \text{Diff}_h[f] \text{ by the monotonicity of the integral} \\ &\qquad\qquad \text{since } \text{Diff}_h[f] \ge 0 \\ &= \int_{\cap_{k=1}^{\infty} O_k} \text{Diff}_h[f] \\ &= \lim_{n\to\infty}\left(\int_{\cap_{k=1}^{n} O_k} \text{Diff}_h[f]\right) \text{ by Continuity of the Integral} \\ &\le \frac{\varepsilon}{2} \text{ since } \cap_{k=1}^{n} O_k \text{ is open and (26}') \text{ applies.}\end{aligned}$$

So establishing (26) yields (25) and the desired result.

Theorem 6.9 (continued 5)

Proof (continued). Choose $\delta > 0$ so that the definition of absolute continuity of f on $[a, b+1]$ is satisfied for $\varepsilon/2$ (where f is extended by setting $f(x) = f(b)$ for $x \in (b, b+1]$ as on page 113). By the change of variables used to get (14) on page 113 (and established in the proof of Corollary 6.4)

$$\begin{aligned}\int_u^v \text{Diff}_h[f] &= \text{Av}_h[f(v)] - \text{Av}[f(u)] = \frac{1}{h}\int_u^{v+h} f - \frac{1}{h}\int_u^{u+h} f \\ &= \frac{1}{h}\int_0^h f(x+v) - \frac{1}{h}\int_0^h f(x+u) \\ &= \frac{1}{h}\int_0^h (f(x+v) - f(x+u)) = \frac{1}{h}\int_0^h g(t)\,dt\end{aligned}$$

for $0 \le t \le 1$ and $a \le u < v \le b$ and $g(t) = f(v+t) - f(u+t)$ (notice that this is a Riemann integral since f is hypothesized to be continuous).

Theorem 6.9 (continued 6)

Theorem 6.9. Let the function f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\le 1}$ is uniformly integrable over $[a, b]$.

Proof (continued). Therefore if $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of (a, b), then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt$ where $E = \cup_{k=1}^n (c_k, d_k)$ and $g(t) = \sum_{k=1}^n [f(d_k + t) - f(c_k + t)]$ for all $0 \le t \le 1$. If $\sum_{k=1}^n (d_k - c_k) < \delta$, then for $0 \le t \le 1$, $\sum_{k=1}^n ((d_k + t) - (c_k + t)) < \delta$ and $g(t) = \sum_{k=1}^n (f(d_k + t) - f(c_k + t)) < \varepsilon/2$ by the choice of δ (to get the $\varepsilon/2$ in the definition of "f is absolutely continuous"). Then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt < \frac{1}{h} h(\varepsilon/2) = \varepsilon/2$. So (26) is confirmed and $\{\text{Diff}_h[f]\}_{0<h\le 1}$ is uniformly integrable. □