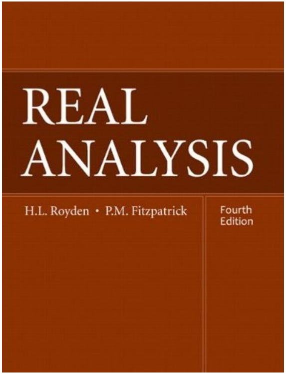
# Real Analysis

**Chapter 6. Differentiation and Integration**

6.4. Absolutely Continuous Functions—Proofs of Theorems



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**Proposition 6.7.** If the function $f$ is Lipschitz on a closed, bounded interval $[a, b]$, then it is absolutely continuous on $[a, b]$.

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# Theorem 6.8

**Theorem 6.8.** Let the function $f$ be absolutely continuous on the closed, bounded interval $[a, b]$. Then $f$ is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

**Proof.** For bounded variation, let $\delta$ correspond to $\varepsilon = 1$ in the definition of absolute continuity of $f$. Let $P$ be a partition of $[a, b]$ into $N$ closed intervals $\{[c_k, d_k]\}_{k=1}^N$, each of length less than $\delta$. Since we chose $\varepsilon = 1$, we have $TV(f_{[c_k,d_k]}) \leq 1$ for $1 \leq k \leq n$.

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Now for the "difference of increasing absolutely continuous functions" part. Since $f$ is of bounded variation, by Lemma 6.5, $f(x) = [f(x) + TV(f_{[a,x]})] - [TV(f_{[a,x]})]$. So the claim follows if $TV(f_{[a,x]})$ is absolutely continuous.

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# Theorem 6.8 (continued)

**Proof (continued).** Let $\varepsilon > 0$. Choose $\delta$ to correspond to $\varepsilon/2$ in the definition of absolute continuity of $f$ on$[a, b]$. Let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of $(a, b)$ for which $\sum_{k=1}^n (d_k - c_k) < \delta$. For $1 \leq k \leq n$, let $P_k$ be a partition of $[c_k, d_k]$. Then $\sum_{k=1}^n V(f_{[c_k,d_k]}, P_k) < \varepsilon/2$ because $V(f_{[c_k,d_k]}, P_k)$ involves a portion of the differences in function values in the definition of absolute continuity and the sum of the differences of the $x$ values in $P_k$ is $d_k - c_k$, so the the choice of $\delta$ we have this sum of differences of function values less than $\varepsilon/2$.

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**Proof (continued).** Let $\varepsilon > 0$. Choose $\delta$ to correspond to $\varepsilon/2$ in the definition of absolute continuity of $f$ on$[a, b]$. Let $\{(c_k, d_k)\}_{k=1}^n$ be a disjoint collection of open subintervals of $(a, b)$ for which $\sum_{k=1}^n (d_k - c_k) < \delta$. For $1 \leq k \leq n$, let $P_k$ be a partition of $[c_k, d_k]$. Then $\sum_{k=1}^n V(f_{[c_k,d_k]}, P_k) < \varepsilon/2$ because $V(f_{[c_k,d_k]}, P_k)$ involves a portion of the differences in function values in the definition of absolute continuity and the sum of the differences of the $x$ values in $P_k$ is $d_k - c_k$, so the the choice of $\delta$ we have this sum of differences of function values less than $\varepsilon/2$. Letting $P_k$ vary over all partitions of $[c_k, d_k]$ gives that $\sum_{k=1}^n TV(f_{[c_k,d_k]}) \leq \varepsilon/2 < \varepsilon$. We infer from (21) in the proof of Lemma 6.5 (see page 117) that for $1 \leq k \leq n$, $TV(f_{[c_k,d_k]}) = TV(f_{[a,d_k]}) - TV(f_{[a,c_k]})$. Hence, if $\sum_{i=1}^n (d_k - c_k) < \delta$ then $\sum_{k=1}^n |TV(f_{[a,d_k]}) - TV(f_{[a,c_k]})| = \sum_{k=1}^n TV(f_{[c_k,d_k]}) \leq \varepsilon/2 < \varepsilon$. So the total variation function $TV(f_{[a,x]})$ is absolutely continuous. □

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# Theorem 6.9

**Theorem 6.9.** Let the function $f$ be continuous on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

**Proof.** First, assume $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$. Let $\varepsilon > 0$.

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**Proof (continued).** So by the choice of $\delta$,

$$\sum_{k=1}^{n} |\mathrm{Av}_h[f(d_k)] = \mathrm{Av}_h[f(c_k)]| = \int_E |\mathrm{Diff}_h[f]| < \varepsilon/2 \qquad (*)$$

for all $0 < h \leq 1$. Since $f$ is continuous then $\lim_{h\to 0^+} \mathrm{Av}_h[f] = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f = f(x)$ (as in Calculus 1), so taking a limit as $h \to 0^+$ in $(*)$ we get $\sum_{k=1}^{n} |f(d_k) - f(c_k)| < \varepsilon/2 < \varepsilon$. Hence, $f$ is absolutely continuous on $[a, b]$.

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# Theorem 6.9 (continued 2)

**Proof (continued).** Therefore the divided differences (WLOG) are nonnegative: $\text{Diff}_h[f] = (f(x+h) - f(x))/h \geq 0$. Let $\varepsilon > 0$. To show uniform integrability, we must find $\delta > 0$ such that for each measurable subset $E$ of $(a, b)$,

$$\int_E \text{Diff}_h[f] < \varepsilon \text{ if } m(E) < \varepsilon \text{ and } 0 < h \leq 1. \quad (25)$$

By Theorem 2.11, measurable set $E$ is contained in a $G_\delta$ set $G$ for which $m(G \setminus E) = 0$.

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# Theorem 6.9 (continued 3)

**Proof (continued).** Below we will show that there is $\delta > 0$ such that for $\{(c_k, d_k)\}_{k=1}^n$ a finite disjoint collection of open subintervals of $(a, b)$:

$$\int_E \text{Diff}_h[f] < \varepsilon/2 \text{ if } m(E) < \delta \text{ where } E = \cup_{k=1}^n (c_k, d_k) \text{ and } 0 < h \leq 1. \quad (26)$$

If (26) is established, then for $E$ an open set with $E = \cup_{k=1}^\infty (c_k, d_k)$ we have by continuity of the integral (Theorem 4.21) that is $m(E) < \delta$ and $0 < h \leq 1$ then

$$\int_E \text{Diff}_h[f] = \int_{\cup_{k=1}^\infty (c_k, d_k)} \text{Diff}_h[f] = \lim_{n\to\infty} \left( \int_{\cup_{k=1}^n (c_k, d_k)} \text{Diff}_n[f] \right) \leq \frac{\varepsilon}{2}. \quad (26')$$

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Now with (26′) established, if $G = \cap_{k=1}^\infty O_k$ is a $G_\delta$ set where $m(G \setminus E) = 0$, then. . .

# Theorem 6.9 (continued 3)

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If (26) is established, then for $E$ an open set with $E = \dot\cup_{k=1}^\infty (c_k, d_k)$ we have by continuity of the integral (Theorem 4.21) that is $m(E) < \delta$ and $0 < h \leq 1$ then

$$\int_E \mathrm{Diff}_h[f] = \int_{\cup_{k=1}^\infty (c_k,d_k)} \mathrm{Diff}_h[f] = \lim_{n\to\infty} \left( \int_{\cup_{k=1}^n (c_k,d_k)} \mathrm{Diff}_n[f] \right) \leq \frac{\varepsilon}{2}. \quad (26')$$

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# Theorem 6.9 (continued 4)

**Proof (continued).**

$$\begin{aligned}
\int_E \text{Diff}_h[f] &\leq \int_G \text{Diff}_h[f] \text{ by the monotonicity of the integral} \\
&\qquad\qquad\qquad\qquad \text{since } \text{Diff}_h[f] \geq 0 \\
&= \int_{\cap_{k=1}^{\infty} O_k} \text{Diff}_h[f] \\
&= \lim_{n\to\infty} \left( \int_{\cap_{k=1}^{n} O_k} \text{Diff}_h[f] \right) \text{ by Continuity of the Integral} \\
&\leq \frac{\varepsilon}{2} \text{ since } \cap_{k=1}^{n} O_k \text{ is open and } (26') \text{ applies.}
\end{aligned}$$

So establishing (26) yields (25) and the desired result.

# Theorem 6.9 (continued 5)

**Proof (continued).** Choose $\delta > 0$ so that the definition of absolute continuity of $f$ on $[a, b+1]$ is satisfied for $\varepsilon/2$ (where $f$ is extended by setting $f(x) = f(b)$ for $x \in (b, b+1]$ as on page 113). By the change of variables used to get (14) on page 113 (and established in the proof of Corollary 6.4)

$$\begin{aligned}
\int_u^v \text{Diff}_h[f] &= \text{Av}_h[f(v)] - \text{Av}[f(u)] = \frac{1}{h}\int_u^{v+h} f - \frac{1}{h}\int_u^{u+h} f \\
&= \frac{1}{h}\int_0^h f(x+v) - \frac{1}{h}\int_0^h f(x+u) \\
&= \frac{1}{h}\int_0^h (f(x+v) - f(x+u)) = \frac{1}{h}\int_0^h g(t)\,dt
\end{aligned}$$

for $0 \le t \le 1$ and $a \le u < v \le b$ and $g(t) = f(v+t) - f(u+t)$ (notice that this is a Riemann integral since $f$ is hypothesized to be continuous).

# Theorem 6.9 (continued 5)

**Proof (continued).** Choose $\delta > 0$ so that the definition of absolute continuity of $f$ on $[a, b+1]$ is satisfied for $\varepsilon/2$ (where $f$ is extended by setting $f(x) = f(b)$ for $x \in (b, b+1]$ as on page 113). By the change of variables used to get (14) on page 113 (and established in the proof of Corollary 6.4)

$$\begin{aligned} \int_u^v \mathrm{Diff}_h[f] &= \mathrm{Av}_h[f(v)] - \mathrm{Av}[f(u)] = \frac{1}{h}\int_u^{v+h} f - \frac{1}{h}\int_u^{u+h} f \\ &= \frac{1}{h}\int_0^h f(x+v) - \frac{1}{h}\int_0^h f(x+u) \\ &= \frac{1}{h}\int_0^h (f(x+v) - f(x+u)) = \frac{1}{h}\int_0^h g(t)\,dt \end{aligned}$$

for $0 \le t \le 1$ and $a \le u < v \le b$ and $g(t) = f(v+t) - f(u+t)$ (notice that this is a Riemann integral since $f$ is hypothesized to be continuous).

# Theorem 6.9 (continued 6)

**Theorem 6.9.** Let the function $f$ be continuous on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

**Proof (continued).** Therefore if $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of $(a, b)$, then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt$ where $E = \cup_{k=1}^n (c_k, d_k)$ and $g(t) = \sum_{k=1}^n [f(d_k + t) - f(c_k + t)]$ for all $0 \leq t \leq 1$. If $\sum_{k=1}^n (d_k - c_k) < \delta$, then for $0 \leq t \leq 1$, $\sum_{k=1}^n ((d_k + t) - (c_k + t)) < \delta$ and $g(t) = \sum_{k=1}^n (f(d_k + t) - f(c_k + t)) < \varepsilon/2$ by the choice of $\delta$ (to get the $\varepsilon/2$ in the definition of "$f$ is absolutely continuous").

# Theorem 6.9 (continued 6)

**Theorem 6.9.** Let the function $f$ be continuous on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

**Proof (continued).** Therefore if $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of $(a, b)$, then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt$ where $E = \cup_{k=1}^n (c_k, d_k)$ and $g(t) = \sum_{k=1}^n [f(d_k + t) - f(c_k + t)]$ for all $0 \leq t \leq 1$. If $\sum_{k=1}^n (d_k - c_k) < \delta$, then for $0 \leq t \leq 1$, $\sum_{k=1}^n ((d_k + t) - (c_k + t)) < \delta$ and $g(t) = \sum_{k=1}^n (f(d_k + t) - f(c_k + t)) < \varepsilon/2$ by the choice of $\delta$ (to get the $\varepsilon/2$ in the definition of "$f$ is absolutely continuous"). Then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt < \frac{1}{h} h(\varepsilon/2) = \varepsilon/2$. So (26) is confirmed and $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable. □

# Theorem 6.9 (continued 6)

**Theorem 6.9.** Let the function $f$ be continuous on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

**Proof (continued).** Therefore if $\{(c_k, d_k)\}_{k=1}^n$ is a disjoint collection of open subintervals of $(a, b)$, then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt$ where $E = \cup_{k=1}^n (c_k, d_k)$ and $g(t) = \sum_{k=1}^n [f(d_k + t) - f(c_k + t)]$ for all $0 \leq t \leq 1$. If $\sum_{k=1}^n (d_k - c_k) < \delta$, then for $0 \leq t \leq 1$, $\sum_{k=1}^n ((d_k + t) - (c_k + t)) < \delta$ and $g(t) = \sum_{k=1}^n (f(d_k + t) - f(c_k + t)) < \varepsilon/2$ by the choice of $\delta$ (to get the $\varepsilon/2$ in the definition of "$f$ is absolutely continuous"). Then $\int_E \text{Diff}_h[f] = \frac{1}{h}\int_0^h g(t)\,dt < \frac{1}{h}h(\varepsilon/2) = \varepsilon/2$. So (26) is confirmed and $\{\text{Diff}_h[f]\}_{0<h\leq 1}$ is uniformly integrable. □