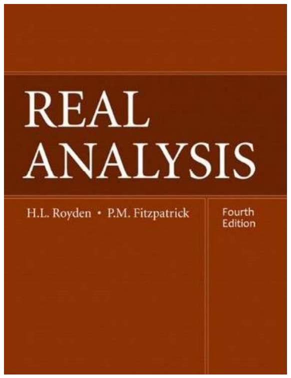
Real Analysis

Chapter 6. Differentiation and Integration

6.5. Integrating Derivatives: Differentiating Indefinite Integrals—Proofs of Theorems



Proposition 6.11

Theorem 6.11. A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

Proof. First, suppose f is absolutely continuous on $[a, b]$. Then for each $x \in (a, b]$, f is absolutely continuous over $[a, x] \subseteq [a, b]$ and so, by the Fundamental Theorem of Lebesgue Calculus Part 1 (Theorem 6.10), with $[a, b]$ replaced with $[a, x]$, $\int_a^x f' = f(x) - f(a)$ or $f(x) = f(a) + \int_a^x f'$. Thus f is the indefinite integral of f' over $[a, b]$.

Second, suppose that f is the indefinite integral over $[a, b]$ of g. For a disjoint collection $\{(a_k, b_k\}_{k=1}^n$ of open intervals in (a, b), if we define $E = \cup_{k=1}^n (a_k, b_k)$ then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \left(f(a) + \int_a^{b_k} g \right) - \left(f(a) + \int_a^{a_k} g \right) \right|$$

since f is the indefinite integral of g

Proposition 6.11 (continued 1)

Proof (continued).

$$\begin{aligned} \sum_{k=1}^n |f(b_k) - f(a_k)| &= \sum_{k=1}^n \left| \int_{a_k}^{b_k} g \right| \text{ by the additivity of integration} \\ &\le \sum_{k=1}^n \left(\int_{a_k}^{b_k} |g| \right) \text{ by the Integral Comparison Test} \\ &= \int_E |g| \text{ by additivity.} \qquad (32) \end{aligned}$$

Let $\varepsilon > 0$. Since $|g|$ is integrable over $[a, b]$, by the definition of indefinite integral, according to Proposition 4.23, there is $\delta > 0$ such that $\int_E |g| < \varepsilon$ if $E \subseteq [a, b]$ is measurable and $m(E) < \delta$. So if $\{(a_k, b_k)\}_{k=1}^n$ is a collection of open intervals with $\sum_{k=1}^n (b_k - a_k) < \delta$, then (32) implies that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$ and so the definition of absolute continuity is satisfied and f is absolutely continuous. □

Corollary 6.12

Corollary 6.12. Let the function f be monotone on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if

$$\int_a^b f' = f(b) - f(a).$$

Proof. By the Fundamental Theorem of Lebesgue Calculus Part 2, the absolute of continuity of f implies the integral equality holds.

Now suppose f is increasing and the integral equality holds. Let $x \in [a, b]$. By additivity of integrals

$$0 = \int_a^b f' - [f(b) - f(a)] = \left\{ \int_a^x f' - [f(x) - f(a)] \right\} + \left\{ \int_x^b f' - [f(b) - f(x)] \right\}. \qquad (33')$$

Corollary 6.12(continued)

Corollary 6.12. Let the function f be monotone on the closed, bounded interval $[a,b]$. Then f is absolutely continuous on $[a,b]$ if and only if

$$\int_a^b f' = f(b) - f(a).$$

Proof (continued). Since f is increasing, then Corollary 6.4 gives $\int_a^x f' \le f(x) - f(a)$ and $\int_x^b f' \le f(b) - f(x)$ or $\int_a^x f' - [f(x) - f(a)] \le 0$ and $\int_x^b f' - [f(b) - f(x)] \le 0$. So (33′) implies that the sum of two nonpositive numbers is zero, and hence the two numbers must be zero. Therefore, $\int_a^x f' = f(x) - f(a)$ or $f(x) = f(a) + \int_a^x f'$ for all $x \in [a,b]$, and so f is the indefinite integral of f' (which exists a.e. on $[a,b]$ by Lebesgue's Theorem). By Theorem 6.11, f is absolutely continuous on $[a,b]$. □

Lemma 6.13

Lemma 6.13. Let f be integrable over the closed bounded interval $[a,b]$. Then $f(x) = 0$ for almost all $x \in [a,b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a,b]$.

Proof. "Clearly" the first condition implies the second.

Suppose the integral condition holds. We claim that $\int_E f = 0$ for all measurable sets $E \subseteq [a,b]$. Of course this holds for all open sets since an open set is a countable disjoint union of open intervals and integration is countably additive (Theorem 4.20). As in the proof of Theorem 6.9, any G_δ set can be written as the limit of a countable descending collection of open sets and the continuity of integration (Theorem 4.21) implies that $\int_G f = 0$ for any G_δ set G. But every measurable set E is of the form $G \setminus E_0$ where G is G_δ and $m(E_0) = 0$ by Theorem 2.11(ii). So $\int_E f = \int_G f + \int_{E \setminus G} f = 0 + 0 = 0$ and the integral equality is verified for all measurable sets E.

Lemma 6.13 (continued)

Lemma 6.13. Let f be integrable over the closed bounded interval $[a,b]$. Then $f(x) = 0$ for almost all $x \in [a,b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a,b]$.

Proof (continued). Now define $E^+ = \{x \in [a,b] \mid f(x) \ge 0\}$ and $E^- = \{x \in [a,b] \mid f(x) \le 0\}$. These are measureable subsets of $[a,b]$ and therefore we have $\int_a^b f^+ = \int_{E^+} f = 0$ and $\int_a^b (-f^-) = -\int_{B^-} f = 0$. Since f^+ and f^- are nonnegative, by Proposition 4.9, a nonnegative integrable function with zero integral must vanish a.e. on its domain. Thus f^+ and f^- vanish a.e. on $[a,b]$ and hence $f = 0$ a.e. on $[a,b]$. □