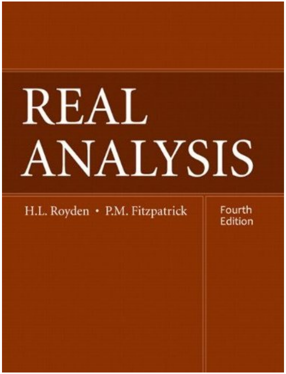
Real Analysis

Chapter 6. Differentiation and Integration

6.5. Integrating Derivatives: Differentiating Indefinite Integrals—Proofs of Theorems



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Second, suppose that f is the indefinite integral over $[a, b]$ of g. For a disjoint collection $\{(a_k, b_k\}_{k=1}^n$ of open intervals in (a, b), if we define $E = \cup_{k=1}^n (a_k, b_k)$ then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \left(f(a) + \int_a^{b_k} g \right) - \left(f(a) + \int_a^{a_k} g \right) \right|$$

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Proof (continued).

$$\begin{aligned}\sum_{k=1}^{n} |f(b_k) - f(a_k)| &= \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right| \text{ by the additivity of integration} \\ &\leq \sum_{k=1}^{n} \left(\int_{a_k}^{b_k} |g| \right) \text{ by the Integral Comparison Test} \\ &= \int_E |g| \text{ by additivity.} \qquad (32)\end{aligned}$$

Let $\varepsilon > 0$. Since $|g|$ is integrable over $[a, b]$, by the definition of indefinite integral, according to Proposition 4.23, there is $\delta > 0$ such that $\int_E |g| < \varepsilon$ if $E \subseteq [a, b]$ is measurable and $m(E) < \delta$.

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$$\int_a^b f' = f(b) - f(a).$$

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$$0 = \int_a^b f' - [f(b) - f(a)] = \left\{ \int_a^x f' - [f(x) - f(a)] \right\} + \left\{ \int_x^b f' - [f(b) - f(x)] \right\}. \qquad (33')$$

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