Chapter 6. Differentiation and Integration
6.5. Integrating Derivatives: Differentiating Indefinite Integrals—Proofs of Theorems
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Theorem 6.11. A function $f$ on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

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Second, suppose that $f$ is the indefinite integral over $[a, b]$ of $g$. For a disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in $(a, b)$, if we define $E = \bigcup_{k=1}^n (a_k, b_k)$ then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \left( f(a) + \int_{a_k}^{b_k} g \right) - \left( f(a) + \int_a^{a_k} g \right) \right|$$

since $f$ is the indefinite integral of $g$.
Proposition 6.11

**Theorem 6.11.** A function $f$ on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

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$$

since $f$ is the indefinite integral of $g$.
Proposition 6.11 (continued 1)

Proof (continued).

\[ \sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right| \text{ by the additivity of integration} \]

\[ \leq \sum_{k=1}^{n} \left( \int_{a_k}^{b_k} |g| \right) \text{ by the Integral Comparison Test} \]

\[ = \int_{E} |g| \text{ by additivity.} \quad (32) \]

Let \( \varepsilon > 0 \). Since \( |g| \) is integrable over \([a, b]\), by the definition of indefinite integral, according to Proposition 4.23, there is \( \delta > 0 \) such that \( \int_{E} |g| < \varepsilon \) if \( E \subseteq [a, b] \) is measurable and \( m(E) < \delta \).
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Proof (continued).

\[
\sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right| \quad \text{by the additivity of integration}
\]

\[
\leq \sum_{k=1}^{n} \left( \int_{a_k}^{b_k} |g| \right) \quad \text{by the Integral Comparison Test}
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\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} g \right| \quad \text{by the additivity of integration}
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Let \( \varepsilon > 0 \). Since \( |g| \) is integrable over \([a, b]\), by the definition of indefinite integral, according to Proposition 4.23, there is \( \delta > 0 \) such that \( \int_{E} |g| < \varepsilon \) if \( E \subseteq [a, b] \) is measurable and \( m(E) < \delta \). So if \( \{(a_k, b_k)\}_{k=1}^{n} \) is a collection of open intervals with \( \sum_{k=1}^{n} (b_k - a_k) < \delta \), then (32) implies that \( \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon \) and so the definition of absolute continuity is satisfied and \( f \) is absolutely continuous. \( \Box \)
Corollary 6.12

**Corollary 6.12.** Let the function $f$ be monotone on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if

$$\int_a^b f' = f(b) - f(a).$$

**Proof.** By the Fundamental Theorem of Lebesgue Calculus Part 2, the absolute of continuity of $f$ implies the integral equality holds.
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Now suppose $f$ is increasing and the integral equality holds. Let $x \in [a, b]$.
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Now suppose $f$ is increasing and the integral equality holds. Let $x \in [a, b]$. By additivity of integrals

$$
0 = \int_a^b f' - [f(b) - f(a)] = \left\{ \int_a^x f' - [f(x) - f(a)] \right\}
$$

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+ \left\{ \int_x^b f' - [f(b) - f(x)] \right\}.
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Proof (continued). Since $f$ is increasing, then Corollary 6.4 gives

$$
\int_a^x f' \leq f(x) - f(a) \quad \text{and} \quad \int_x^b f' \leq f(b) - f(x) \text{ or } \int_a^x f' - [f(x) - f(a)] \leq 0
$$

and

$$
\int_x^b f' - [f(b) - f(x)] \leq 0.
$$

So (33’) implies that the sum of two nonpositive numbers is zero, and hence the two numbers must be zero. Therefore, $\int_a^x f' = f(x) - f(a)$ or $f(x) = f(a) + \int_a^x f'$ for all $x \in [a, b]$, and so $f$ is the indefinite integral of $f'$ (which exists a.e. on $[a, b]$ by Lebesgue’s Theorem).
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So (33') implies that the sum of two nonpositive numbers is zero, and hence the two numbers must be zero. Therefore,

\[
\int_a^x f' = f(x) - f(a) \quad \text{or} \quad f(x) = f(a) + \int_a^x f' \quad \text{for all} \quad x \in [a, b],
\]

and so \( f \) is the indefinite integral of \( f' \) (which exists a.e. on \([a, b]\) by Lebesgue’s Theorem).

By Theorem 6.11, \( f \) is absolutely continuous on \([a, b]\).
Corollary 6.12. Let the function $f$ be monotone on the closed, bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if

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Therefore, $\int_a^x f' = f(x) - f(a)$ or $f(x) = f(a) + \int_a^x f'$ for all $x \in [a, b]$, and so $f$ is the indefinite integral of $f'$ (which exists a.e. on $[a, b]$ by Lebesgue's Theorem). By Theorem 6.11, $f$ is absolutely continuous on $[a, b]$. \qed
Lemma 6.13. Let $f$ be integrable over the closed bounded interval $[a, b]$. Then $f(x) = 0$ for almost all $x \in [a, b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a, b]$.

Proof. “Clearly” the first condition implies the second.
Lemma 6.13. Let \( f \) be integrable over the closed bounded interval \([a, b]\). Then \( f(x) = 0 \) for almost all \( x \in [a, b] \) if and only if \( \int_{x_1}^{x_2} f = 0 \) for all \( (x_1, x_2) \subseteq [a, b] \).

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Suppose the integral condition holds. We claim that \( \int_E f = 0 \) for all measurable sets \( E \subseteq [a, b] \). Of course this holds for all open sets since an open set is a countable disjoint union of open intervals and integration is countably additive (Theorem 4.20).
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Lemma 6.13. Let $f$ be integrable over the closed bounded interval $[a, b]$. Then $f(x) = 0$ for almost all $x \in [a, b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a, b]$.

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Proof (continued). Now define $E^+ = \{x \in [a, b] \mid f(x) \geq 0\}$ and $E^- = \{x \in [a, b] \mid f(x) \leq 0\}$. These are measurable subsets of $[a, b]$ and therefore we have $\int_a^b f^+ = \int_{E^+} f = 0$ and $\int_a^b (-f^-) = -\int_{E^-} f = 0$. Since $f^+$ and $f^-$ are nonnegative, by Proposition 4.9, a nonnegative integrable function with zero integral must vanish a.e. on its domain. Thus $f^+$ and $f^-$ vanish a.e. on $[a, b]$ and hence $f = 0$ a.e. on $[a, b]$. \qed
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