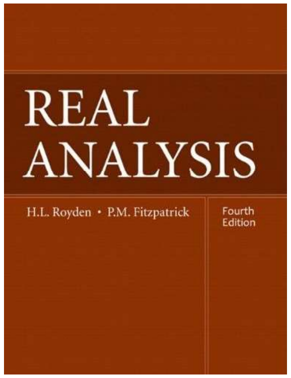
Real Analysis

Chapter 6. Differentiation and Integration
6.6. Convex Functions—Proofs of Theorems



Proposition 6.15

Proposition 6.15. If φ is differentiable on (a, b) and its derivative φ' is increasing, then φ is convex. In particular, if φ'' exists on (a, b) and $\varphi'' \geq 0$ on (a, b), then φ is convex.

Proof. Let $x_1 < x_2$ be in (a, b) and let $x \in (x_1, x_2)$. Since φ is differentiable, then the Mean Value Theorem applies to φ on intervals $[x_1, x]$ and $[x, x_2]$. So choose $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ for which $\varphi'(c_1) = (\varphi(x) - \varphi(x_1))/(x - x_1)$ and $\varphi'(c_2) = (\varphi(x_2) - \varphi(x))/(x_2 - x)$. Since φ' is increasing then

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} = \varphi'(c_1) \leq \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

So (39) holds and hence φ is convex. □

The Chordal Slope Lemma

The Chordal Slope Lemma. Let φ be convex on (a, b). If $x_1 < x < x_2$ are in (a, b), then for points $P_1 = (x_1, \varphi(x_1))$, $P = (x, \varphi(x))$, and $P_2 = (x_2, \varphi(x_2))$ we have

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

That is, the slope of $\overline{P_1P}$ is less than or equal to the slope of $\overline{P_1P_2}$, which is less than or equal to the slope of $\overline{PP_2}$.

Proof. From (38′) we have $(x_2 - x_1)\varphi(x) \leq x_2\varphi(x_1) - x_1\varphi(x_1) + x_1\varphi(x_1) - x\varphi(x_1) + (x - x_1)\varphi(x_2) = (x_2 - x_1)\varphi(x_1) + (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2)$ if and only if $(x_2 - x_1)(\varphi(x) - \varphi(x_1)) \leq (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) = (x - x_1)(\varphi(x_2) - \varphi(x_1))$ if and only if $\dfrac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \dfrac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1}$.

The Chordal Slope Lemma (continued)

The Chordal Slope Lemma. Let φ be convex on (a, b). If $x_1 < x < x_2$ are in (a, b), then for points $P_1 = (x_1, \varphi(x_1))$, $P = (x, \varphi(x))$, and $P_2 = (x_2, \varphi(x_2))$ we have

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$

That is, the slope of $\overline{P_1P}$ is less than or equal to the slope of $\overline{P_1P_2}$, which is less than or equal to the slope of $\overline{PP_2}$.

Proof (continued). Also from (38′) we have $-(x_2 - x)\varphi(x_1) \leq (x - x_1)\varphi(x_2) - (x_2 - x_1)\varphi(x) = x\varphi(x_2) - x_2\varphi(x_2) + x_2\varphi(x_2) - x_1\varphi(x_2) - (x_2 - x_1)\varphi(x) = -(x_2 - x)\varphi(x_2) + (x_2 - x_1)(\varphi(x_2) - \varphi(x))$ if and only if $(x_2 - x)(\varphi(x_2) - \varphi(x_1)) \leq (x_2 - x_1)(\varphi(x_2) - \varphi(x))$ if and only if $\dfrac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \dfrac{\varphi(x_2) - \varphi(x)}{x_2 - x}$. □

Corollary 6.17

Corollary 6.17. Let φ be a convex function on (a, b). Then φ is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval $[c, d]$ of (a, b).

Proof. According to Lemma 6.16, for $c \le u < v \le d$,

$$\begin{aligned} \varphi'(c^+) &\le \varphi'(u^+) \text{ by Lemma 6.16 applied to } u = c \text{ and } v = u \\ &\le \frac{\varphi(v) - \varphi(u)}{v - u} \text{ by Lemma 6.16} \\ &\le \varphi'(v^-) \text{ by Lemma 6.16} \\ &\le \varphi'(d^-) \text{ by Lemma 6.16 applied to } u = v \text{ and } v = d. \end{aligned}$$

Therefore, with $M = \max\{|\varphi'(c^+)|, |\varphi'(d^-)|\}$ (which exist and are finite by Lemma 6.16) we have $|\varphi(v) - \varphi(u)| \le M|v - u|$ for all $u, v \in [c, d]$. So φ is Lipschitz on $[u, v]$. A Lipschitz function on a closed, bounded interval is absolutely continuous on this interval by Proposition 6.7. □

Theorem 6.18

Theorem 6.18. Let φ be a convex function on (a, b). Then φ is differentiable except at a countable number of points and its derivative φ' is an increasing function.

Proof. By the inequalities of Lemma 6.16, we have that the functions mapping $x \mapsto f'(x^-)$ and $x \mapsto f'(x^+)$ are increasing real-valued functions on (a, b). By Theorem 6.1, these two functions are continuous except at a countable number of points. So, except on a countable subset $\mathcal{C}$ of (a, b), both the left-hand and right-hand derivatives of φ are continuous. Let $x_0 \in (a, b) \setminus \mathcal{C}$. Choose a sequence $\{x_n\}$ in (a, b) (and possibly in $\mathcal{C}$) of points greater than x_0 that converges to x_0. With $u = x_0$ and $v = x_n$ in Lemma 6.16 we have $\varphi'(x_0^-) \le \varphi'(x_0^+) \le \varphi'(x_n^-)$. Now let $n \to \infty$ so that $x_n \to x_0$. Since the left-hand derivative is continuous at x_0 then $\varphi'(x_n^-) \to \varphi'(x_0^-)$. So it must be that $\varphi'(x_0^-) = \varphi'(x_0^+)$ and φ is differentiable at x_0. So φ is differentiable on $(a, b) \setminus \mathcal{C}$.

Theorem 6.18 (continued)

Theorem 6.18. Let φ be a convex function on (a, b). Then φ is differentiable except at a countable number of points and its derivative φ' is an increasing function.

Proof (continued). To show that φ' is an increasing function on $(a, b) \setminus \mathcal{C}$, let $u < v$ belong to $(a, b) \setminus \mathcal{C}$. Then by Lemma 6.16, $\varphi'(u) \le \dfrac{\varphi(u) - \varphi(v)}{u - v} \le \varphi'(v)$, and so φ' is increasing on $(a, b) \setminus \mathcal{C}$. □

Lemma

Lemma. Let φ be a convex function on (a, b) and let x_0 belong to (a, b). Then there is a supporting line at x_0 for the graph of φ for every slope between $\varphi'(x_0^-)$ and $\varphi'(x_0^+)$.

Proof. Let $y = m(x - x_0) + \varphi(x_0)$ where $\varphi'(x_0^-) \le m \le \varphi'(x_0^+)$. Then with $u = x_0$ and $v = x \in (x_0, b)$ we have by Lemma 6.16 that $\varphi'(x_0^+) \le (\varphi(x) - \varphi(x_0))/(x - x_0)$ or $\varphi'(x_0^+)(x - x_0) \le \varphi(x) - \varphi(x_0)$. So $m(x - x_0) \le \varphi'(x_0^+)(x - x_0) \le \varphi(x) - \varphi(x_0)$ and $m(x - x_0) + \varphi(x_0) \le \varphi(x)$ for $x \in (x_0, b)$. With $u = x \in (a, x_0)$ and $v = x_0$ we have by Lemma 6.16 that $(\varphi(x_0) - \varphi(x))/(x_0 - x) \le \varphi'(x_0^-)$ or $\varphi(x_0) - \varphi(x) \le \varphi'(x_0^-)(x_0 - x) \le m(x_0 - x)$ or $m(x - x_0) \le \varphi(x) - \varphi(x_0)$, and $m(x - x_0) + \varphi(x_0) \le \varphi(x)$ for $x \in (a, x_0)$. Hence $y = m(x - x_0) + \varphi(x_0)$ is a supporting line at x_0 provided $\varphi'(x_0^-) \le m \le \varphi'(x_0^+)$. □

Jensen's Inequality

Jensen's Inequality. Let φ be a convex function on $(-\infty, \infty)$, f an integrable function over $[0,1]$, and $\varphi \circ f$ also integrable over $[0,1]$. Then

$$\varphi\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 (\varphi \circ f)(x)\,dx.$$

Proof. Let $\alpha = \int_0^1 f(x)\,dx$. Choose m to lie between the left-hand and right-hand derivative of φ at α. Then by "Lemma" $y = m(t-\alpha) + \varphi(\alpha)$ is a supporting line at $(\alpha, \varphi(\alpha))$ for the graph of φ. Hence $\varphi(t) \geq m(t-\alpha) + \varphi(\alpha)$ for all $t \in \mathbb{R}$. Since f is integrable over $[0,1]$, then f is finite a.e. on $[0,1]$ by Proposition 4.15, and therefore with $t = f(x)$ we have $\varphi(f(x)) \geq m(f(x)-\alpha) + \varphi(\alpha)$ for almost all $x \in [0,1]$.

Jensen's Inequality (continued)

Jensen's Inequality. Let φ be a convex function on $(-\infty, \infty)$, f an integrable function over $[0,1]$, and $\varphi \circ f$ also integrable over $[0,1]$. Then

$$\varphi\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 (\varphi \circ f)(x)\,dx.$$

Proof (continued). Since f and $\varphi \circ f$ are hypothesized to be integrable over $[0,1]$, then by monotonicity of the integral (Theorem 4.17) we have

$$\int_0^1 \varphi(f(x))\,dx \geq \int_0^1 (m(f(x)-\alpha) + \varphi(\alpha))\,dx$$
$$= m\left(\int_0^1 f(x)\,dx - \int_0^1 \alpha\,dx\right) + \int_0^1 \varphi(\alpha)\,dx$$
$$= m\left(\int_0^1 f(x)\,dx - \alpha\right) + \varphi(\alpha) = \varphi(\alpha) = \varphi\left(\int_0^1 f(x)\,dx\right).$$

□