Chapter 6. Differentiation and Integration

6.6. Convex Functions—Proofs of Theorems
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Proposition 6.15. If $\varphi$ is differentiable on $(a, b)$ and its derivative $\varphi'$ is increasing, then $\varphi$ is convex. In particular, if $\varphi''$ exists on $(a, b)$ and $\varphi'' \geq 0$ on $(a, b)$, then $\varphi$ is convex.

Proof. Let $x_1 < x_2$ be in $(a, b)$ and let $x \in (x_1, x_2)$. Since $\varphi$ is differentiable, then the Mean Value Theorem applies to $\varphi$ on intervals $[x_1, x]$ and $[x, x_2]$. 
Proposition 6.15. If $\varphi$ is differentiable on $(a, b)$ and its derivative $\varphi'$ is increasing, then $\varphi$ is convex. In particular, if $\varphi''$ exists on $(a, b)$ and $\varphi'' \geq 0$ on $(a, b)$, then $\varphi$ is convex.

Proof. Let $x_1 < x_2$ be in $(a, b)$ and let $x \in (x_1, x_2)$. Since $\varphi$ is differentiable, then the Mean Value Theorem applies to $\varphi$ on intervals $[x_1, x]$ and $[x, x_2]$. So choose $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ for which $\varphi'(c_1) = (\varphi(x) - \varphi(x_1))/(x - x_1)$ and $\varphi'(c_2) = (\varphi(x_2) - \varphi(x))/(x_2 - x)$. 
Proposition 6.15. If φ is differentiable on (a, b) and its derivative φ' is increasing, then φ is convex. In particular, if φ'' exists on (a, b) and φ'' ≥ 0 on (a, b), then φ is convex.

Proof. Let x_1 < x_2 be in (a, b) and let x ∈ (x_1, x_2). Since φ is differentiable, then the Mean Value Theorem applies to φ on intervals [x_1, x] and [x, x_2]. So choose c_1 ∈ (x_1, x) and c_2 ∈ (x, x_2) for which

\[ φ'(c_1) = \frac{φ(x) - φ(x_1)}{x - x_1} \quad \text{and} \quad φ'(c_2) = \frac{φ(x_2) - φ(x)}{x_2 - x}. \]

Since φ' is increasing then

\[ \frac{φ(x) - φ(x_1)}{x - x_1} = φ'(c_1) \leq φ'(c_2) = \frac{φ(x_2) - φ(x)}{x_2 - x}. \]

So (39) holds and hence φ is convex. \[ \square \]
**Proposition 6.15.** If $\varphi$ is differentiable on $(a, b)$ and its derivative $\varphi'$ is increasing, then $\varphi$ is convex. In particular, if $\varphi''$ exists on $(a, b)$ and $\varphi'' \geq 0$ on $(a, b)$, then $\varphi$ is convex.

**Proof.** Let $x_1 < x_2$ be in $(a, b)$ and let $x \in (x_1, x_2)$. Since $\varphi$ is differentiable, then the Mean Value Theorem applies to $\varphi$ on intervals $[x_1, x]$ and $[x, x_2]$. So choose $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ for which $\varphi'(c_1) = (\varphi(x) - \varphi(x_1))/(x - x_1)$ and $\varphi'(c_2) = (\varphi(x_2) - \varphi(x))/(x_2 - x)$. Since $\varphi'$ is increasing then

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} = \varphi'(c_1) \leq \varphi'(c_2) = \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.$$ 

So (39) holds and hence $\varphi$ is convex. \qed
The Chordal Slope Lemma

The Chordal Slope Lemma. Let \( \varphi \) be convex on \((a, b)\). If \( x_1 < x < x_2 \) are in \((a, b)\), then for points \( P_1 = (x_1, \varphi(x_1)) \), \( P = (x, \varphi(x)) \), and \( P_2 = (x_2, \varphi(x_2)) \) we have

\[
\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.
\]

That is, the slope of \( \overline{P_1P} \) is less than or equal to the slope of \( \overline{P_1P_2} \), which is less than or equal to the slope of \( \overline{PP_2} \).

Proof. From (38') we have

\[
(x_2 - x_1)\varphi(x) \leq x_2\varphi(x_1) - x_1\varphi(x_1) + x_1\varphi(x_1) - x\varphi(x_1) + (x - x_1)\varphi(x_2) = (x_2 - x_1)\varphi(x_1) + (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) \text{ if and only if}
\]
The Chordal Slope Lemma. Let \( \varphi \) be convex on \((a, b)\). If \( x_1 < x < x_2 \) are in \((a, b)\), then for points \( P_1 = (x_1, \varphi(x_1)) \), \( P = (x, \varphi(x)) \), and \( P_2 = (x_2, \varphi(x_2)) \) we have

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\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.
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\[
(x_2 - x_1)\varphi(x) \leq x_2\varphi(x_1) - x_1\varphi(x_1) + x_1\varphi(x_1) - x\varphi(x_1) + (x - x_1)\varphi(x_2) = (x_2 - x_1)\varphi(x_1) + (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2)
\]

if and only if

\[
(x_2 - x_1)(\varphi(x) - \varphi(x_1)) \leq (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) = (x - x_1)(\varphi(x_2) - \varphi(x_1))
\]

if and only if

\[
\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1}.
\]
The Chordal Slope Lemma. Let \( \varphi \) be convex on \((a, b)\). If \( x_1 < x < x_2 \) are in \((a, b)\), then for points \( P_1 = (x_1, \varphi(x_1)) \), \( P = (x, \varphi(x)) \), and \( P_2 = (x_2, \varphi(x_2)) \) we have

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\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.
\]

That is, the slope of \( \overline{P_1P} \) is less than or equal to the slope of \( \overline{P_1P_2} \), which is less than or equal to the slope of \( \overline{PP_2} \).

**Proof.** From \((38')\) we have

\[
(x_2 - x_1)\varphi(x) \leq x_2\varphi(x_1) - x_1\varphi(x_1) + x_1\varphi(x_1) - x\varphi(x_1) + (x - x_1)\varphi(x_2) = (x_2 - x_1)\varphi(x_1) + (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) \text{ if and only if}
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(x_2 - x_1)(\varphi(x) - \varphi(x_1)) \leq (x_1 - x)\varphi(x_1) + (x - x_1)\varphi(x_2) = (x - x_1)(\varphi(x_2) - \varphi(x_1)) \text{ if and only if } \frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1}.
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The Chordal Slope Lemma. Let $\varphi$ be convex on $(a, b)$. If $x_1 < x < x_2$ are in $(a, b)$, then for points $P_1 = (x_1, \varphi(x_1))$, $P = (x, \varphi(x))$, and $P_2 = (x_2, \varphi(x_2))$ we have

$$
\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.
$$

That is, the slope of $P_1P$ is less than or equal to the slope of $P_1P_2$, which is less than or equal to the slope of $PP_2$.

Proof (continued). Also from (38’) we have

$$
-(x_2 - x)\varphi(x_1) \leq (x - x_1)\varphi(x_2) - (x_2 - x_1)\varphi(x) = x\varphi(x_2) - x_2\varphi(x_2) + x_2\varphi(x_2) - x_1\varphi(x_2) - (x_2 - x_1)\varphi(x) = -(x_2 - x)\varphi(x_2) + (x_2 - x_1)(\varphi(x_2) - \varphi(x))
$$

if and only if $(x_2 - x)(\varphi(x_2) - \varphi(x_1)) \leq (x_2 - x_1)(\varphi(x_2) - \varphi(x))$ if and only if $\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}$. \hfill \Box
The Chordal Slope Lemma. Let \( \varphi \) be convex on \((a, b)\). If \( x_1 < x < x_2 \) are in \((a, b)\), then for points \( P_1 = (x_1, \varphi(x_1)) \), \( P = (x, \varphi(x)) \), and \( P_2 = (x_2, \varphi(x_2)) \) we have

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\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x}.
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That is, the slope of \( \overline{P_1P} \) is less than or equal to the slope of \( \overline{P_1P_2} \), which is less than or equal to the slope of \( \overline{PP_2} \).

Proof (continued). Also from \((38')\) we have

\[-(x_2 - x)\varphi(x_1) \leq (x - x_1)\varphi(x_2) - (x_2 - x_1)\varphi(x) = x\varphi(x_2) - x_2\varphi(x_2) + x_2\varphi(x_2) - x_1\varphi(x_2) - (x_2 - x_1)\varphi(x) = -(x_2 - x)\varphi(x_2) + (x_2 - x_1)(\varphi(x_2) - \varphi(x))\]

if and only if \((x_2 - x)(\varphi(x_2) - \varphi(x_1)) \leq (x_2 - x_1)(\varphi(x_2) - \varphi(x))\) if and

only if \(\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \).
Corollary 6.17

**Corollary 6.17.** Let \( \varphi \) be a convex function on \((a, b)\). Then \( \varphi \) is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval \([c, d]\) of \((a, b)\).

**Proof.** According to Lemma 6.16, for \( c \leq u < v \leq d \),

\[
\varphi'(c^+) \leq \varphi'(u^+) \quad \text{by Lemma 6.16 applied to } u = c \text{ and } v = u \\
\leq \frac{\varphi(v) - \varphi(u)}{v - u} \quad \text{by Lemma 6.16} \\
\leq \varphi'(v^-) \quad \text{by Lemma 6.16} \\
\leq \varphi'(d^-) \quad \text{by Lemma 6.16 applied to } u = v \text{ and } v = d.
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\]
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\leq \varphi'(v^-) \quad \text{by Lemma 6.16}
\]
\[
\leq \varphi'(d^-) \quad \text{by Lemma 6.16 applied to } u = v \text{ and } v = d.
\]

Therefore, with \( M = \max\{|\varphi'(c^+)|, |\varphi'(d^-)|\} \) (which exist and are finite by Lemma 6.16) we have \(|\varphi(v) - \varphi(u)| \leq M|v - u|\) for all \( u, v \in [c, d] \). So \( \varphi \) is Lipschitz on \([u, v]\).
Corollary 6.17

**Corollary 6.17.** Let $\varphi$ be a convex function on $(a, b)$. Then $\varphi$ is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval $[c, d]$ of $(a, b)$.

**Proof.** According to Lemma 6.16, for $c \leq u < v \leq d$,

\[
\varphi'(c^+) \leq \varphi'(u^+) \text{ by Lemma 6.16 applied to } u = c \text{ and } v = u
\]
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\leq \frac{\varphi(v) - \varphi(u)}{v - u} \text{ by Lemma 6.16}
\]
\[
\leq \varphi'(v^-) \text{ by Lemma 6.16}
\]
\[
\leq \varphi'(d^-) \text{ by Lemma 6.16 applied to } u = v \text{ and } v = d.
\]

Therefore, with $M = \max\{|\varphi'(c^+)|, |\varphi'(d^-)|\}$ (which exist and are finite by Lemma 6.16) we have $|\varphi(v) - \varphi(u)| \leq M|v - u|$ for all $u, v \in [c, d]$. So $\varphi$ is Lipschitz on $[u, v]$. A Lipschitz function on a closed, bounded interval is absolutely continuous on this interval by Proposition 6.7.
Corollary 6.17. Let \( \varphi \) be a convex function on \((a, b)\). Then \( \varphi \) is Lipschitz, and therefore absolutely continuous, on each closed, bounded subinterval \([c, d]\) of \((a, b)\).

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\leq \frac{\varphi(v) - \varphi(u)}{v - u} \text{ by Lemma 6.16}
\]
\[
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\]
\[
\leq \varphi'(d^-) \text{ by Lemma 6.16 applied to } u = v \text{ and } v = d.
\]

Therefore, with \( M = \max\{|\varphi'(c^+)|, |\varphi'(d^-)|\} \) (which exist and are finite by Lemma 6.16) we have \(|\varphi(v) - \varphi(u)| \leq M|v - u|\) for all \( u, v \in [c, d]\). So \( \varphi \) is Lipschitz on \([u, v]\). A Lipschitz function on a closed, bounded interval is absolutely continuous on this interval by Proposition 6.7.
Theorem 6.18. Let $\varphi$ be a convex function on $(a, b)$. Then $\varphi$ is differentiable except at a countable number of points and its derivative $\varphi'$ is an increasing function.

Proof. By the inequalities of Lemma 6.16, we have that the functions mapping $x \mapsto f'(x^-)$ and $x \mapsto f'(x^+)$ are increasing real-valued functions on $(a, b)$. By Theorem 6.1, these two functions are continuous except at a countable number of points. So, except on a countable subset $C$ of $(a, b)$, both the left-hand and right-hand derivatives of $\varphi$ are continuous.
Theorem 6.18. Let $\varphi$ be a convex function on $(a, b)$. Then $\varphi$ is differentiable except at a countable number of points and its derivative $\varphi'$ is an increasing function.

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Theorem 6.18. Let \( \varphi \) be a convex function on \((a, b)\). Then \( \varphi \) is differentiable except at a countable number of points and its derivative \( \varphi' \) is an increasing function.

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Theorem 6.18. Let \( \varphi \) be a convex function on \((a, b)\). Then \( \varphi \) is differentiable except at a countable number of points and its derivative \( \varphi' \) is an increasing function.

Proof. By the inequalities of Lemma 6.16, we have that the functions mapping \( x \mapsto f'(x^-) \) and \( x \mapsto f'(x^+) \) are increasing real-valued functions on \((a, b)\). By Theorem 6.1, these two functions are continuous except at a countable number of points. So, except on a countable subset \( C \) of \((a, b)\), both the left-hand and right-hand derivatives of \( \varphi \) are continuous. Let \( x_0 \in (a, b) \setminus C \). Choose a sequence \( \{x_n\} \) in \((a, b)\) (and possibly in \( C \)) of points greater than \( x_0 \) that converges to \( x_0 \). With \( u = x_0 \) and \( v = x_n \) in Lemma 6.16 we have \( \varphi'(x_0^-) \leq \varphi'(x_0^+) \leq \varphi'(x_n^-) \). Now let \( n \to \infty \) so that \( x_n \to x_0 \). Since the left-hand derivative is continuous at \( x_0 \) then \( \varphi'(x_n^-) \to \varphi'(x_0^-) \). So it must be that \( \varphi'(x_0^-) = \varphi'(x_0^+) \) and \( \varphi \) is differentiable at \( x_0 \). So \( \varphi \) is differentiable on \((a, b) \setminus C\).
Theorem 6.18. Let \( \varphi \) be a convex function on \((a, b)\). Then \( \varphi \) is differentiable except at a countable number of points and its derivative \( \varphi' \) is an increasing function.

Proof (continued). To show that \( \varphi' \) is an increasing function on \((a, b) \setminus C\), let \( u < v \) belong to \((a, b)\). Then by Lemma 6.16,
\[
\varphi'(u) \leq \frac{\varphi(u) - \varphi(v)}{u - v} \leq \varphi'(v),
\]
and so \( \varphi' \) is increasing on \((a, b) \setminus C\). \( \square \)
Lemma. Let $\varphi$ be a convex function on $(a, b)$ and let $x_0$ belong to $(a, b)$. Then there is a supporting line at $x_0$ for the graph of $\varphi$ for every slope between $\varphi'(x_0^-)$ and $\varphi'(x_0^+)$. 

Proof. Let $y = m(x - x_0) + \varphi(x_0)$ where $\varphi'(x_0^-) \leq m \leq \varphi'(x_0^+)$. Then with $u = x_0$ and $\nu = x \in (x_0, b)$ we have by Lemma 6.16 that $\varphi'(x_0^+) \leq (\varphi(x) - \varphi(x_0))/(x - x_0)$ or $\varphi'(x_0^+)(x - x_0) \leq \varphi(x) - \varphi(x_0)$.
Lemma. Let \( \varphi \) be a convex function on \( (a, b) \) and let \( x_0 \) belong to \( (a, b) \). Then there is a supporting line at \( x_0 \) for the graph of \( \varphi \) for every slope between \( \varphi'(x_0^-) \) and \( \varphi'(x_0^+) \).

Proof. Let \( y = m(x - x_0) + \varphi(x_0) \) where \( \varphi'(x_0^-) \leq m \leq \varphi'(x_0^+) \). Then with \( u = x_0 \) and \( v = x \in (x_0, b) \) we have by Lemma 6.16 that
\[
\varphi'(x_0^+) \leq (\varphi(x) - \varphi(x_0))/(x - x_0) \quad \text{or} \quad \varphi'(x_0^+)(x - x_0) \leq \varphi(x) - \varphi(x_0).
\]
So
\[
m(x - x_0) \leq \varphi'(x_0^+)(x - x_0) \leq \varphi(x) - \varphi(x_0) \quad \text{and}
\]
\[
m(x - x_0) + \varphi(x_0) \leq \varphi(x) \quad \text{for} \quad x \in (x_0, b).
\]
With \( u = x \in (a, x_0) \) and \( v = x_0 \) we have by Lemma 6.16 that
\[
(\varphi(x_0) - \varphi(x))/(x_0 - x) \leq \varphi'(x_0^-) \quad \text{or}
\]
\[
\varphi(x_0) - \varphi(x) \leq \varphi'(x_0^-)(x_0 - x) \leq m(x_0 - x) \quad \text{or}
\]
\[
m(x - x_0) \leq \varphi(x) - \varphi(x_0), \quad \text{and} \quad m(x - x_0) + \varphi(x_0) \leq \varphi(x) \quad \text{for} \quad x \in (a, x_0).
Lemma. Let $\varphi$ be a convex function on $(a, b)$ and let $x_0$ belong to $(a, b)$. Then there is a supporting line at $x_0$ for the graph of $\varphi$ for every slope between $\varphi'(x_0^-)$ and $\varphi'(x_0^+)$. 

Proof. Let $y = m(x - x_0) + \varphi(x_0)$ where $\varphi'(x_0^-) \leq m \leq \varphi'(x_0^+)$. Then with $u = x_0$ and $v = x \in (x_0, b)$ we have by Lemma 6.16 that 

$$
\varphi'(x_0^+) \leq (\varphi(x) - \varphi(x_0))/(x - x_0) \text{ or } \varphi'(x_0^+)(x - x_0) \leq \varphi(x) - \varphi(x_0). \text{ So } \quad m(x - x_0) \leq \varphi'(x_0^+)(x - x_0) \leq \varphi(x) - \varphi(x_0) \text{ and }
$$

$$
m(x - x_0) + \varphi(x_0) \leq \varphi(x) \text{ for } x \in (x_0, b). \text{ With } u = x \in (a, x_0) \text{ and } v = x_0 \text{ we have by Lemma 6.16 that } (\varphi(x_0) - \varphi(x))/(x_0 - x) \leq \varphi'(x_0^-) \text{ or }
$$

$$
\varphi(x_0) - \varphi(x) \leq \varphi'(x_0^-)(x_0 - x) \leq m(x_0 - x) \text{ or }
$$

$$
m(x - x_0) \leq \varphi(x) - \varphi(x_0), \text{ and } m(x - x_0) + \varphi(x_0) \leq \varphi(x) \text{ for } x \in (a, x_0). \text{ Hence } y = m(x - x_0) + \varphi(x_0) \text{ is a supporting line at } x_0 \text{ provided } \varphi'(x_0^-) \leq m \leq \varphi'(x_0^+). \blacksquare
Lemma. Let \( \varphi \) be a convex function on \((a, b)\) and let \( x_0 \) belong to \((a, b)\). Then there is a supporting line at \( x_0 \) for the graph of \( \varphi \) for every slope between \( \varphi'(x^-_0) \) and \( \varphi'(x^+_0) \).

Proof. Let \( y = m(x - x_0) + \varphi(x_0) \) where \( \varphi'(x^-_0) \leq m \leq \varphi'(x^+_0) \). Then with \( u = x_0 \) and \( v = x \in (x_0, b) \) we have by Lemma 6.16 that
\[
\varphi'(x^+_0) \leq (\varphi(x) - \varphi(x_0))/(x - x_0) \text{ or } \varphi'(x^+_0)(x - x_0) \leq \varphi(x) - \varphi(x_0). \]
So
\[
m(x - x_0) \leq \varphi'(x^+_0)(x - x_0) \leq \varphi(x) - \varphi(x_0) \text{ and }
m(x - x_0) + \varphi(x_0) \leq \varphi(x) \text{ for } x \in (x_0, b). \]
With \( u = x \in (a, x_0) \) and \( v = x_0 \) we have by Lemma 6.16 that
\[
(\varphi(x_0) - \varphi(x))/(x_0 - x) \leq \varphi'(x^-_0) \text{ or } \varphi(x_0) - \varphi(x) \leq \varphi'(x^-_0)(x_0 - x) \leq m(x_0 - x) \text{ or }
m(x - x_0) \leq \varphi(x) - \varphi(x_0), \text{ and } m(x - x_0) + \varphi(x_0) \leq \varphi(x) \text{ for } x \in (a, x_0). \]
Hence \( y = m(x - x_0) + \varphi(x_0) \) is a supporting line at \( x_0 \) provided
\[
\varphi'(x^-_0) \leq m \leq \varphi'(x^+_0). \]
Jensen’s Inequality

**Jensen’s Inequality.** Let \( \varphi \) be a convex function on \((-\infty, \infty)\), \( f \) an integrable function over \([0, 1]\), and \( \varphi \circ f \) also integrable over \([0, 1]\). Then

\[
\varphi \left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 (\varphi \circ f)(x) \, dx.
\]

**Proof.** Let \( \alpha = \int_0^1 f(x) \, dx \). Choose \( m \) to lie between the left-hand and right-hand derivative of \( \varphi \) at \( \alpha \). Then by “Lemma” \( y = m(t - \alpha) + \varphi(\alpha) \) is a supporting line at \((\alpha, \varphi(\alpha))\) for the graph of \( \varphi \).
Jensen’s Inequality

Jensen’s Inequality. Let $\varphi$ be a convex function on $(-\infty, \infty)$, $f$ an integrable function over $[0, 1]$, and $\varphi \circ f$ also integrable over $[0, 1]$. Then

$$
\varphi \left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 (\varphi \circ f)(x) \, dx.
$$

Proof. Let $\alpha = \int_0^1 f(x) \, dx$. Choose $m$ to lie between the left-hand and right-hand derivative of $\varphi$ at $\alpha$. Then by “Lemma” $y = m(t - \alpha) + \varphi(\alpha)$ is a supporting line at $(\alpha, \varphi(\alpha))$ for the graph of $\varphi$. Hence $\varphi(t) \geq m(t - \alpha) + \varphi(\alpha)$ for all $t \in \mathbb{R}$. Since $f$ is integrable over $[0, 1]$, then $f$ is finite a.e. on $[0, 1]$ by Proposition 4.15, and therefore with $t = f(x)$ we have $\varphi(f(x)) \geq m(f(x) - \alpha) + \varphi(\alpha)$ for almost all $x \in [0, 1]$. 
Jensen’s Inequality. Let $\varphi$ be a convex function on $(-\infty, \infty)$, $f$ an integrable function over $[0, 1]$, and $\varphi \circ f$ also integrable over $[0, 1]$. Then

$$\varphi \left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 (\varphi \circ f)(x) \, dx.$$  

Proof. Let $\alpha = \int_0^1 f(x) \, dx$. Choose $m$ to lie between the left-hand and right-hand derivative of $\varphi$ at $\alpha$. Then by “Lemma” $y = m(t - \alpha) + \varphi(\alpha)$ is a supporting line at $(\alpha, \varphi(\alpha))$ for the graph of $\varphi$. Hence

$$\varphi(t) \geq m(t - \alpha) + \varphi(\alpha) \text{ for all } t \in \mathbb{R}.$$  

Since $f$ is integrable over $[0, 1]$, then $f$ is finite a.e. on $[0, 1]$ by Proposition 4.15, and therefore with $t = f(x)$ we have $\varphi(f(x)) \geq m(f(x) - \alpha) + \varphi(\alpha)$ for almost all $x \in [0, 1]$. 


Jensen’s Inequality (continued)

**Jensen’s Inequality.** Let \( \varphi \) be a convex function on \((-\infty, \infty)\), \( f \) an integrable function over \([0, 1]\), and \( \varphi \circ f \) also integrable over \([0, 1]\). Then

\[
\varphi \left( \int_0^1 f(x) \, dx \right) \leq \int_0^1 (\varphi \circ f)(x) \, dx.
\]

**Proof (continued).** Since \( f \) and \( \varphi \circ f \) are hypothesized to be integrable over \([0, 1]\), then by monotonicity of the integral (Theorem 4.17) we have

\[
\int_0^1 \varphi(f(x)) \, dx \geq \int_0^1 (m(f(x) - \alpha) + \varphi(\alpha)) \, dx
\]

\[
= m \left( \int_0^1 f(x) \, dx - \int_0^1 \alpha \, dx \right) + \int_0^1 \varphi(\alpha) \, dx
\]

\[
= m \left( \int_0^1 f(x) \, dx - \alpha \right) + \varphi(\alpha) = \varphi(\alpha) = \varphi \left( \int_0^1 f(x) \, dx \right).
\]
