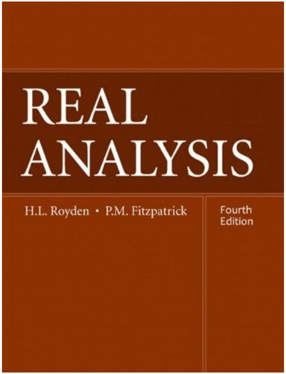
Real Analysis

Chapter 7. The L^p Spaces: Completeness and Approximation

7.2. The Inequalities of Young, Holder, and Minkowski—Proofs of Theorems



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Young's Inequality

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$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Consider $g(x) = x^p/p + 1/q - x$. Then $g'(x) = x^{p-1} - 1$ and so $g'(x) < 0$ when $x \in (0,1)$ and $g'(x) > 0$ when $x \in (1,\infty)$. Therefore g has a minimum at $x = 1$ (of 0). So $g(x) \geq 0$ for $x > 0$. Therefore $x \leq x^p/p + 1/q$ for $x > 0$. With $x = a/b^{q-1} > 0$ we have

$$\begin{aligned} \frac{a}{b^{q-1}} &\leq \frac{1}{p}\left(\frac{a}{b^{q-1}}\right)^p + \frac{1}{q} \\ &= \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q} \text{ since } p(q-1) = q, \end{aligned}$$

or $ab \leq a^p/p + b^q/q$. □

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Theorem 7.1. Hölder's Inequality

Theorem 7.1. Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$, then fg is integrable over E and

$$\int_E |fg| \leq \|f\|_p \|g\|_q.$$

This is *Hölder's Inequality*. Moreover, if $f \neq 0$, then the function

$$f^* = \begin{cases} \|f\|_p^{1-p} \operatorname{sgn}(f)|f|^{p-1} & \text{if } p > 1 \\ \operatorname{sgn}(f) & \text{if } p = 1 \end{cases}$$

is an element of $L^q(E)$,

$$\int_E ff^* = \|f\|_p$$

and $\|f^*\|_q = 1$.

Theorem 7.1. Hölder's Inequality (continued 1)

Proof. If $p = 1$ and $q = \infty$ then
$\|fg\|_1 = \int_E |fg| \leq \|g\|_\infty \int_E |f| = \|f\|_1 \|g\|_\infty$, and Hölder's Inequality holds. With $p = 1$, $f^* = \text{sgn}(f)$ and so $ff^* = |f|$ and
$\int_E ff^* = \int_E |f| = \|f\|_1 = \|f\|_p$. Also,
$\|f^*\|_q = \|f^*\|_\infty = \text{ess sup}_{x \in E} |f^*(x)| = 1$.

Consider $p > 1$. The results are trivial if $f = 0$ or $g = 0$. "It is clear" that if Hölder's Inequality is true for "normalized" $f/\|f\|_p$ and $g/\|g\|_q$, then it is true for all f and g (as appropriate). So without loss of generality, assume $\|f\|_p = \|g\|_q = 1$. Since $|f|^p$ and $|g|^q$ are integrable over E, then f and g are finite a.e. on E (Proposition 4.13).

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Proof. If $p = 1$ and $q = \infty$ then $\|fg\|_1 = \int_E |fg| \leq \|g\|_\infty \int_E |f| = \|f\|_1 \|g\|_\infty$, and Hölder's Inequality holds. With $p = 1$, $f^* = \text{sgn}(f)$ and so $ff^* = |f|$ and $\int_E ff^* = \int_E |f| = \|f\|_1 = \|f\|_p$. Also, $\|f^*\|_q = \|f^*\|_\infty = \text{ess sup}_{x \in E} |f^*(x)| = 1$.

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$$\|fg\|_1 = \int_E |fg| \leq \frac{1}{p} \int_E |f|^p + \frac{1}{q} \int_E |g|^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q.$$

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Proof (continued). Finally,

$$ff^* = f\|f\|_p^{1-p}\text{sgn}(f)|f|^{p-1} = \|f\|_p^{1-p}|f|^p,$$

$$\int_E ff^* = \|f\|_p^{1-p}\int_E |f|^p = \|f\|_p^{1-p}\|f\|_p^p = \|f\|_p,$$

and

$$\begin{aligned}\|f^*\|_q &= \left\{\int_E \left|\|f\|_p^{1-p}\text{sgn}(f)|f|^{p-1}\right|^q\right\}^{1/q} \\ &= \left\{\int_E |f|^p\right\}^{1/q} \text{ since } q(p-1) = p \\ &= \left(\left\{\int_E |f|^p\right\}^{1/p}\right)^{p/q} = (1)^{p/q} = 1.\end{aligned}$$

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Let E be measurable and $1 \leq p \leq \infty$. If f and g belong to $L^p(E)$, then $f + g \in L^p(E)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. We have already seen that the Triangle Inequality holds for $p = 1$ (in Example 7.1.B) and for $p = \infty$ (see Example 7.1.C). So, without loss of generality, we suppose $p \in (1, \infty)$. We saw in Example 7.1.A that for all $a, b \in \mathbb{R}$ we have $|a + b|^p \leq 2^p\{|a|^p + |b|^p\}$, and so by monotonicity of integration (Theorem 4.10), $f + g \in L^p(E)$.

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$$\begin{aligned} \|f + g\|_p &= \int_E (f + g)(f + g)^* \text{ by Theorem 7.1} \\ &= \int_E f(f + g)^* + \int_E g(f + g)^*. \end{aligned}$$

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Proof (continued). Now $\int_E |f(f+g)^*| \leq \|f\|_p \|(f+g)^*\|_q$ by Hölder's Inequality and $f(f+g)^* \leq |f(f+g)^*|$ on E, so by the Integral Comparison Test (Proposition 4.16),

$$\int_E f(f+g)^* \leq \left| \int_E f(f+g)^* \right| \leq \int_E |f(f+g)^*| \leq \|f\|_p \|(f+g)^*\|_q.$$

Similarly $\int_E g(f+g)^* \leq \|g\|_p \|(f+g)^*\|_q$. Hence

$$\begin{aligned} \|f+g\|_p &= \int_E f(f+g)^* + \int_E g(f+g)^* \\ &\leq \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \\ &= \|f\|_p + \|g\|_q \text{ since } \|(f+g)^*\|_q = 1 \text{ by Hölder's Inequality} \\ &\quad \text{(the "Moreover" part).} \end{aligned}$$

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Proof. Assume $p_2 < \infty$. Define $p = p_2/p_1 > 1$ and let q be the conjugate of p. Let $f \in L^{p_2}(E)$. Then $|f|^{p_1} \in L^p(E)$ and $g = \chi_E \in L^q(E)$ since $m(E) < \infty$.

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$$\int_E |f|^{p_1} \leq m(E)(\text{ess sup}(f))^{p_1} = m(E)\|f\|_\infty^{p_1} < \infty$$

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