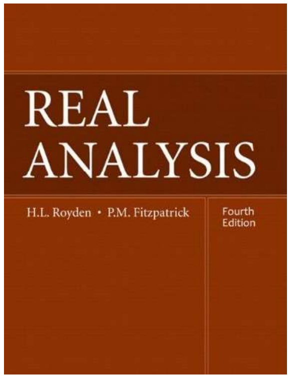
Real Analysis

Chapter 7. The L^p Spaces: Completeness and Approximation

7.3. L^p is Complete: The Riesz-Fischer Theorem—Proofs of Theorems



Proposition 7.4

Proposition 7.4. Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Proof. Let $\{f_n\} \to f$ in X. Then $\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\| \le \|f_n - f\| + \|f - f_m\|$ for all $m, n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all values of the index greater than N, we have $\|f_n - f\| < \varepsilon/2$. Then for all $m, n > N$, we have $\|f_m - f_n\| \le \|f_n - f\| + \|f_m - f\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Now, let $\{f_n\}$ be a Cauchy sequence in X with subsequence $\{f_{n_k}\}$ which converges to f in X. Let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, choose $N_1 \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon/2$ for all $m, n \ge N_1$. Since $\{f_{n_k}\}$ converges to f there is $N_2 \in \mathbb{N}$ such that if $n_k \ge N_2$ then $\|f_{n_k} - f\| < \varepsilon/2$. So for $n \ge \max\{N_1, N_2\}$ we have

$$\|f_n - f\| = \|(f_n - f_{n_k}) + (f_{n_k} - f)\| \le \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{f_n\} \to f$. □

Proposition 7.5

Proposition 7.5. Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore every Cauchy sequence has a rapidly Cauchy subsequence.

Proof. Let $\{f_n\}$ be rapidly Cauchy in X with $\{\varepsilon_k\}_{k=1}^\infty$ as described above. Then

$$\|f_{n+k} - f_n\| = \left\| \sum_{j=n}^{n+k-1} (f_{j+1} - f_j) \right\| \le \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \le \sum_{j=n}^{n+k-1} \varepsilon_j^2 \le \sum_{j=n}^{\infty} \varepsilon_j^2.$$

Since $\sum_{k=1}^\infty \varepsilon_k$ converges, then $\sum_{k=1}^\infty \varepsilon_k^2$ converges ($\varepsilon_k \to 0$, eventually $\varepsilon_k^2 \le \varepsilon_k$, and the Comparison Test). Therefore for n "sufficiently large," $\sum_{j=n}^\infty \varepsilon_j^2$ is "small" hence $\{f_n\}$ is a Cauchy sequence.

Now assume $\{f_n\}$ is Cauchy in X. For any f_{n_k} we may find, by the Cauchy property, $f_{n_{k+1}}$ such that $\|f_{n_{k+1}} - f_{n_k}\| \le (1/2)^k \equiv \varepsilon_k^2$. Then $\sum_{k=1}^\infty \varepsilon_k = \sum_{k=1}^\infty (1/\sqrt{2})^k$ converges (geometric series). So $\{f_{n_k}\}$ is rapidly Cauchy. □

Theorem 7.6

Theorem 7.6. Let E be measurable and $1 \le p \le \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the L^p norm and pointwise a.e. on E to a function in $L^p(E)$.

Proof. The case $p = \infty$ is Exercise 7.33. Assume $1 \le p < \infty$. Let $\{f_n\}$ be a rapidly Cauchy sequence in $L^p(E)$. Without loss of generality, each f_n is finite valued. Choose $\{\varepsilon_k\}_{k=1}^\infty$ as described above. Then $\|f_{k+1} - f_k\|_p \le \varepsilon_k^2$ and so

$$\int_E |f_{k+1} - f_k|^p \le \varepsilon_k^{2p} \qquad (*)$$

for $k \in \mathbb{N}$. Fix index k. Now for $x \in E$, we have $|f_{k+1}(x) - f_k(x)| \ge \varepsilon_k$ if and only if $|f_{k+1}(x) - f_k(x)|^p \ge \varepsilon_k^p$, so by Chebychev's Inequality (see Section 4.3. The Lebesgue Integral of a Measurable Nonnegative Function) we have...

Theorem 7.6 (continued 1)

Proof (continued). . . .

$$m(\{x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k\}) = m(\{x \in E \mid |f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p\})$$

$$\leq \frac{1}{\varepsilon_k^p} \int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^p \text{ by } (*).$$

Since $p \geq 1$, the series $\sum_{k=1}^\infty \varepsilon_k^p$ converges ($\varepsilon_k \to 0$, so eventually $\varepsilon_k^p < \varepsilon_k$ and by the Comparison Test). By the Borel-Cantelli Lemma (see Section 2.5. Continuity and the Borel-Cantelli Lemma), since $\sum_{k=1}^\infty m(\{x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k\}) \leq \sum_{k=1}^\infty \varepsilon_k^p < \infty$, almost all $x \in E$ belong to finitely many of the sets on the left hand side. That is, there is set $E_0 \subset E$ where $m(E_0) = 0$, and for $x \in E \setminus E_0$ we have x in finitely many of the sets on the left hand side. In other words, for each $x \in E \setminus E_0$ there is an index $K(x)$ (think of it as the index of the "last" set containing x) such that $|f_{k+1}(x) - f_k(x)| < \varepsilon_k$ for all $k > K(x)$.

Theorem 7.6 (continued 2)

Proof (continued). Then for all $n \geq K(x)$ and all $k \in \mathbb{N}$ we have

$$|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)| \leq \sum_{j=n}^\infty \varepsilon_j.$$

Since $\sum_{j=1}^\infty \varepsilon_j$ converges, for n sufficiently large, the right hand side of this inequality can be made small, and so the sequence of real numbers (for fixed x) $\{f_k(x)\}$ is Cauchy and therefore convergent. Define $f(x)$ as $\lim_{n\to\infty} f_n(x) = f(x)$. It follows as in the proof of Theorem 7.5 that $\|f_{n+k} - f_n\|_p \leq \sum_{j=n}^\infty \varepsilon_j^2$ or $\int_E |f_{n+k} - f_n|^p \leq \left(\sum_{j=n}^\infty \varepsilon_j^2\right)^p$ for all $n, k \in \mathbb{N}$. When $k \to \infty$, $f_{n+k} \to f$ a.e. on E and so, by Fatou's Lemma, $\int_E \lim_{k\to\infty} |f_{n+k} - f_n|^p = \int_E |f - f_n|^p \leq \underline{\lim}_{k\to\infty} \int |f_{n+k} - f_n|^p \leq \left(\sum_{j=n}^\infty \varepsilon_j^2\right)^p$ for all $n \in \mathbb{N}$. Therefore $f - f_n \in L^p(E)$ and since $f_n \in L^p(E)$, then $f \in L^p(E)$. The right hand side of this inequality can be made arbitrarily small by making n sufficiently large, and so $\{f_n\} \to f$ in $L^p(E)$. So $\{f_n\}$ converges to f in L^p and a.e. pointwise by the construction of f. □

The Riesz-Fischer Theorem

The Riesz-Fischer Theorem. Let E be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in L^p then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on E to f.

Proof. We need to show completeness. Suppose $\{f_n\}$ is a Cauchy sequence in L^p. By Proposition 7.5, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that is rapidly Cauchy. By Theorem 7.6, $\{f_{n_k}\}$ converges to an $f \in L^p(E)$ both with respect to the L^p norm and a.e. pointwise on E. By Proposition 7.4, $\{f_n\}$ converges to f with respect to the L^p norm. □

Theorem 7.7

Theorem 7.7. Let E be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the L^p norm if and only if

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

Proof. Without loss of generality, we assume f and each f_n is real-valued and the convergence is pointwise on E. By Minkowski's Inequality, $\|f_n\|_p = \|f_n - f + f\|_p \leq \|f_n - f\|_p + \|f\|_p$, or $\|f_n\|_p - \|f\|_p \leq \|f_n - f\|_p$. Also $\|f\|_p = \|f - f_n + f_n\|_p \leq \|f_n\|_p + \|f - f_n\|_p$ or $-\|f - f_n\|_p \leq \|f_n\|_p - \|f\|_p$. Therefore $|\|f_n\|_p - \|f\|_p| \leq \|f - f_n\|_p$. So if $\{f_n\} \to f$ with respect to the L^p norm, then $\|f_n\|_p \to \|f\|_p$. To prove the converse, suppose $\|f_n\|_p \to \|f\|_p$ and $\{f_n\} \to f$ pointwise on E.

Theorem 7.7 (continued 1)

Proof (continued). Define $\psi(t) = |t|^p$. Then ψ is "convex" (i.e., concave up since $p \geq 1$) and $\psi((a+b)/2) \leq (\psi(a)+\psi(b))/2$ for all a, b. Hence $0 \leq (|a|^p + |b|^p)/2 - |(a-b)/2|^p$ for all a, b (here, we are using $\psi((a+(-b))/2) \leq (\psi(a)+\psi(-b))/2$). Therefore, for each n, h_n is nonnegative and measurable on E where $h_n(x) = (|f_n(x)|^p + |f(x)|^p)/2 - |f_n(x) - f(x)|^p/2^p$. Then $h_n \to |f|^p$ since $f_n \to f$ pointwise. So by Fatou's Lemma and since $\|f_n\|_p \to \|f\|_p$,

$$\int_E |f|^p \leq \liminf \int_E h_n = \liminf \int_E \left(\frac{|f_n|^p + |f|^p}{2} - \frac{|f_n - f|^p}{2^p} \right)$$

$$= \liminf \left(\int_E \frac{|f_n|^p}{2} \right) + \int_E \frac{|f|^p}{2} - \limsup \left(\int_E \frac{|f_n - f|^p}{2^p} \right)$$

$$= \int_E \frac{|f|^p}{2} + \int_E \frac{|f|^p}{2} - \limsup \int_E \frac{|f_n - f|^p}{2^p} = \int_E |f|^p - \limsup \int_E \frac{|f_n - f|^p}{2^p}.$$

Theorem 7.7 (continued 2)

Theorem 7.7. Let E be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the L^p norm if and only if

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

Proof (continued). Therefore $\limsup \int_E |(f_n - f)/2|^p \leq 0$, or $\limsup \int_E |f_n - f|^p \leq 0$ or $\lim \int_E |f_n - f|^p = 0$ (since $|f_n - f|$ nonnegative) and therefore $\|f_n - f\|_p \to 0$, or $\{f_n\} \to f$ with respect to the L^p norm. □