Proposition 7.4

Proposition 7.4. Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

Proof. Let $\{f_n\}$ converge to $f$ in $X$. Then
$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\|$$
for all $n, m \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all values of the index greater than $N$, we have $\|f_n - f\| < \varepsilon/2$. Then for all $m, n > N$, we have
$$\|f_m - f_n\| \leq \|f_m - f\| + \|f - f_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$  
Now let $\{f_n\}$ be a Cauchy sequence in $X$ with subsequence $\{f_{n_k}\}$ which converges to $f$ in $X$. Let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, choose $N_1 \in \mathbb{N}$ such that $\|f_n - f\| < \varepsilon/2$ for all $m, n \geq N_1$. Since $\{f_{n_k}\}$ converges to $f$ there is $N_2 \in \mathbb{N}$ such that if $n_k \geq N_2$ then $\|f_{n_k} - f\| < \varepsilon/2$. So for $n \geq \max\{N_1, N_2\}$ we have
$$\|f_n - f\| = \|(f_n - f_{n_k}) + (f_{n_k} - f)\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$  
So $\{f_n\} \to f$. \qed

Proposition 7.5

Proposition 7.5. Let $X$ be a normed linear space. Then every rapidly Cauchy sequence in $X$ is Cauchy. Furthermore every Cauchy sequence has a rapidly Cauchy subsequence.

Proof. Let $\{f_n\}$ be rapidly Cauchy in $X$ with $\{\varepsilon_k\}_{k=1}^{\infty}$ as described above. Then
$$\|f_{n+k} - f_n\| = \left\| \sum_{j=n}^{n+k-1} (f_{j+1} - f_j) \right\| \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{n+k-1} \varepsilon_j^2 \leq \sum_{j=n}^{\infty} \varepsilon_j^2.$$  
Since $\sum_{k=1}^{\infty} \varepsilon_k$ converges, then $\sum_{k=1}^{\infty} \varepsilon_k^2$ converges ($\varepsilon_k \to 0$, eventually $\varepsilon_k^2 \leq \varepsilon_k$, and the Comparison Test). Therefore for $n$ “sufficiently large,” $\sum_{j=n}^{\infty} \varepsilon_j^2$ is “small” hence $\{f_n\}$ is a Cauchy sequence.

Now assume $\{f_n\}$ is Cauchy in $X$. For any $f_{n_k}$ we may find, by the Cauchy property, $f_{n_k+1}$ such that $\|f_{n_k+1} - f_{n_k}\| \leq (1/2)^k \varepsilon_k^2$. Then
$$\sum_{k=1}^{\infty} \varepsilon_k = \sum_{k=1}^{\infty} (1/\sqrt{2})^k$$
converges (geometric series). So $\{f_n\}$ is rapidly Cauchy. \qed

Theorem 7.6

Theorem 7.6. Let $E$ be measurable and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p$ norm and pointwise a.e. on $E$ to a function in $L^p(E)$.

Proof. The case $p = \infty$ is Exercise 7.33. Assume $1 \leq p < \infty$. Let $\{f_n\}$ be a rapidly Cauchy sequence in $L^p(E)$. Without loss of generality, each $f_n$ is finite valued. Choose $\{\varepsilon_k\}_{k=1}^{\infty}$ as described above. Then
$$\|f_{k+1} - f_k\|_p \leq \varepsilon_k^2$$
and so
$$\int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^{2p} \quad (*)$$
for $k \in \mathbb{N}$. Fix index $k$. Now for $x \in E$, we have $|f_{k+1}(x) - f_k(x)| \geq \varepsilon_k$ if and only if $|f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p$, so by Chebychev’s Inequality (page 80) we have
Theorem 7.6 (continued)

Proof (continued). 

\[ m(\{x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k\}) = m(\{x \in E \mid |f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p\}) \leq \frac{1}{\varepsilon_k^p} \int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^p \] by (*).

Since \( p > 1 \), the series \( \sum_{k=1}^{\infty} \varepsilon_k^p \) converges (\( \varepsilon_k \to 0 \), so eventually \( \varepsilon_k^p < \varepsilon_k \) and by the Comparison Test). By the Borel-Cantelli Lemma (page 46), since \( \sum_{k=1}^{\infty} m(\{x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k\}) \leq \sum_{k=1}^{\infty} \varepsilon_k^p < \infty \), almost all \( x \in E \) belong to finitely many of the sets on the left hand side. That is, there is set \( E_0 \subset E \) where \( m(E_0) = 0 \), and for \( x \in E \setminus E_0 \) we have \( x \) in finitely many of the sets on the left hand side. In other words, for each \( x \in E \setminus E_0 \) there is an index \( K(x) \) (think of it as the index of the “last” set containing \( x \)) such that \( |f_{k+1}(x) - f_k(x)| < \varepsilon_k^p \) for all \( k > K(x) \).

The Riesz-Fischer Theorem

The Riesz-Fischer Theorem. Let \( E \) be measurable and \( 1 \leq p < \infty \). Suppose \( \{f_n\} \) is a sequence in \( L^p(E) \) that converges pointwise a.e. on \( E \) to \( f \). Then \( \{f_n\} \rightarrow f \) in \( L^p \) if and only if

\[ \lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p, \]

that is \( \|f_n\|_p \to \|f\|_p \).

Proof. We need to show completeness. Suppose \( \{f_n\} \) is a Cauchy sequence in \( L^p \). By Proposition 7.5, there is a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) that is rapidly Cauchy. By Theorem 7.6, \( \{f_{n_k}\} \) converges to an \( f \in L^p(E) \) both with respect to the \( L^p \) norm and a.e. pointwise on \( E \). By Proposition 7.4, \( \{f_n\} \) converges to \( f \) with respect to the \( L^p \) norm.

Theorem 7.7

Theorem 7.7. Let \( E \) be measurable and \( 1 \leq p < \infty \). Suppose \( \{f_n\} \) is a sequence in \( L^p(E) \) that converges pointwise a.e. on \( E \) to \( f \in L^p(E) \). Then \( \{f_n\} \rightarrow f \) with respect to the \( L^p \) norm if and only if

\[ \lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p, \]

that is \( \|f_n\|_p \to \|f\|_p \).

Proof. Without loss of generality, we assume \( f \) and each \( f_n \) is real-valued and the convergence is pointwise on \( E \). By Minkowski’s Inequality, \( \|f_n\|_p = \|f_n - f + f\|_p \leq \|f_n - f\|_p + \|f\|_p \), or \( \|f_n - f\|_p \to \|f\|_p \).

Also \( \|f\|_p = \|f - f_n + f_n\|_p \leq \|f - f_n\|_p + \|f_n\|_p \) or \( \|f - f_n\|_p \leq \|f_n\|_p + \|f\|_p \). Therefore \( \|f_n\|_p \to \|f\|_p \). So if \( \{f_n\} \rightarrow f \) with respect to the \( L^p \) norm, then \( \|f_n\|_p \to \|f\|_p \) or \( \{f_n\} \rightarrow f \) pointwise on \( E \). To prove the converse, suppose \( \|f_n\|_p \to \|f\|_p \) and \( \{f_n\} \rightarrow f \) pointwise on \( E \).
Theorem 7.7 (continued 1)

Proof (continued). Define \( \psi(t) = |t|^p \). Then \( \psi \) is “convex” (i.e., concave up since \( p \geq 1 \)) and \( \psi((a + b)/2) \leq (\psi(a) + \psi(b))/2 \) for all \( a, b \). Hence \( 0 \leq ((a)^p + |b|^p)/2 - ((a - b)^p)/2 \) for all \( a, b \) (here, we are using \( \psi((a + (-b))/2) \leq (\psi(a) + \psi(-b))/2 \)). Therefore, for each \( n, h_n \) is nonnegative and measurable on \( E \) where

\[
h_n(x) = \left( |f_n(x)|^p + |f(x)|^p \right)/2 - |f_n(x) - f(x)|^p/2^p.
\]

Then \( h_n \to |f|^p \) since \( f_n \to f \) pointwise. So by Fatou’s Lemma and since \( \|f_n\|_p \to \|f\|_p \),

\[
\int_E |f|^p \leq \lim inf \int_E h_n = \lim inf \int_E \left( \frac{|f_n|^p + |f|^p}{2} - \frac{|f_n - f|^p}{2^p} \right)
- \lim inf \left( \int_E \frac{|f_n|^p}{2} \right) + \int_E \frac{|f|^p}{2} - \lim sup \left( \int_E \frac{|f_n - f|^p}{2^p} \right)
= \int_E \frac{|f|^p}{2} + \int_E \frac{|f|^p}{2} - \lim sup \int_E \frac{|f_n - f|^p}{2^p} = \int_E |f|^p - \lim sup \int_E \frac{|f_n - f|^p}{2^p}.
\]

Theorem 7.7 (continued 2)

Theorem 7.7. Let \( E \) be measurable and \( 1 \leq p < \infty \). Suppose \( \{f_n\} \) is a sequence in \( L^p(E) \) that converges pointwise a.e. on \( E \) to \( f \in L^p(E) \). Then \( \{f_n\} \to f \) with respect to the \( L^p \) norm if and only if

\[
\lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p;
\]

that is \( \|f_n\|_p \to \|f\|_p \).

Proof (continued). Therefore \( \lim sup \int_E |(f_n - f)/2|^p \leq 0 \), or \( \lim sup \int_E |f_n - f|^p \leq 0 \) or \( \lim \int_E |f_n - f|^p = 0 \) (since \( |f_n - f| nonnegative \)) and therefore \( \|f_n - f\|_p \to 0 \), or \( \{f_n\} \to f \) with respect to the \( L^p \) norm. \( \square \)