Chapter 7. The $L^p$ Spaces: Completeness and Approximation

7.3. $L^p$ is Complete: The Riesz-Fischer Theorem—Proofs of Theorems
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Proposition 7.4.

Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

Proof. Let $\{f_n\} \to f$ in $X$. Then

$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\| \leq \|f_n - f\| + \|f - f_m\|$$

for all $m, n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all values of the index greater than $N$, we have $\|f_n - f\| < \epsilon/2$. Then for all $m, n > N$, we have

$$\|f_m - f_n\| \leq \|f_n - f\| + \|f_m - f\| < \epsilon/2 + \epsilon/2 = \epsilon.$$
Proposition 7.4. Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

**Proof.** Let $\{f_n\} \to f$ in $X$. Then

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$$\|f_m - f_n\| \leq \|f_n - f\| + \|f_m - f\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Now, let $\{f_n\}$ be a Cauchy sequence in $X$ with subsequence $\{f_{n_k}\}$ which converges to $f$ in $X$. Let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, choose $N_1 \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon/2$ for all $m, n \geq N_1$. Since $\{f_{n_k}\}$ converges to $f$ there is $N_2 \in \mathbb{N}$ such that if $n_k \geq N_2$ then $\|f_{n_k} - f\| < \varepsilon/2$. 
Proposition 7.4. Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

**Proof.** Let $\{f_n\} \to f$ in $X$. Then
\[
\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\| \leq \|f_n - f\| + \|f - f_m\| \quad \text{for all } m, n \in \mathbb{N}.
\]
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Now, let $\{f_n\}$ be a Cauchy sequence in $X$ with subsequence $\{f_{n_k}\}$ which converges to $f$ in $X$. Let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, choose $N_1 \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon/2$ for all $m, n \geq N_1$. Since $\{f_{n_k}\}$ converges to $f$ there is $N_2 \in \mathbb{N}$ such that if $n_k \geq N_2$ then $\|f_{n_k} - f\| < \varepsilon/2$. So for $n \geq \max\{N_1, N_2\}$ we have
\[
\|f_n - f\| = \|(f_n - f_{n_k}) + (f_{n_k} - f)\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
So $\{f_n\} \to f$. \qed
Proposition 7.4. Let $X$ be a normed linear space. Then every convergent sequence in $X$ is Cauchy. Moreover, a Cauchy sequence in $X$ converges if it has a convergent subsequence.

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$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\| \leq \|f_n - f\| + \|f - f_m\|$$

for all $m, n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all values of the index greater than $N$, we have $\|f_n - f\| < \varepsilon/2$. Then for all $m, n > N$, we have

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So $\{f_n\} \to f$. 

\[\square\]
**Proposition 7.5.** Let $X$ be a normed linear space. Then every rapidly Cauchy sequence in $X$ is Cauchy. Furthermore every Cauchy sequence has a rapidly Cauchy subsequence.

**Proof.** Let $\{f_n\}$ be rapidly Cauchy in $X$ with $\{\varepsilon_k\}_{k=1}^{\infty}$ as described above. Then

$$
\|f_{n+k} - f_n\| = \left\| \sum_{j=n}^{n+k-1} (f_{j+1} - f_j) \right\| \leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\| \leq \sum_{j=n}^{n+k-1} \varepsilon_j^2 \leq \sum_{j=n}^{\infty} \varepsilon_j^2.
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$$

Since $\sum_{k=1}^{\infty} \varepsilon_k$ converges, then $\sum_{k=1}^{\infty} \varepsilon_k^2$ converges ($\varepsilon_k \to 0$, eventually $\varepsilon_k^2 \leq \varepsilon_k$, and the Comparison Test). Therefore for $n$ “sufficiently large,” $\sum_{j=n}^{\infty} \varepsilon_j^2$ is “small” hence $\{f_n\}$ is a Cauchy sequence.
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Now assume $\{f_n\}$ is Cauchy in $X$. For any $f_{n_k}$ we may find, by the Cauchy property, $f_{n_{k+1}}$ such that $\|f_{n_{k+1}} - f_{n_k}\| \leq (1/2)^k \equiv \varepsilon_k^2$. Then $\sum_{k=1}^{\infty} \varepsilon_k = \sum_{k=1}^{\infty} (1/\sqrt{2})^k$ converges (geometric series). So $\{f_{n_k}\}$ is rapidly Cauchy.
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Proof. Let $\{f_n\}$ be rapidly Cauchy in $X$ with $\{\varepsilon_k\}_{k=1}^\infty$ as described above. Then

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Since $\sum_{k=1}^\infty \varepsilon_k$ converges, then $\sum_{k=1}^\infty \varepsilon_k^2$ converges ($\varepsilon_k \to 0$, eventually $\varepsilon_k^2 \leq \varepsilon_k$, and the Comparison Test). Therefore for $n$ “sufficiently large,” $\sum_{j=n}^\infty \varepsilon_j^2$ is “small” hence $\{f_n\}$ is a Cauchy sequence.

Now assume $\{f_n\}$ is Cauchy in $X$. For any $f_{n_k}$ we may find, by the Cauchy property, $f_{n_{k+1}}$ such that $\|f_{n_{k+1}} - f_{n_k}\| \leq (1/2)^k \equiv \varepsilon_k^2$. Then $\sum_{k=1}^\infty \varepsilon_k = \sum_{k=1}^\infty (1/\sqrt{2})^k$ converges (geometric series). So $\{f_{n_k}\}$ is rapidly Cauchy.

$\square$
Theorem 7.6. Let $E$ be measurable and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p$ norm and pointwise a.e. on $E$ to a function in $L^p(E)$.

Proof. The case $p = \infty$ is Exercise 7.33. Assume $1 \leq p < \infty$. Let $\{f_n\}$ be a rapidly Cauchy sequence in $L^p(E)$. Without loss of generality, each $f_n$ is finite valued.
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$$\|f_{k+1} - f_k\|_p \leq \varepsilon_k^2$$

and so

$$\int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^{2p} \quad (*)$$

for $k \in \mathbb{N}$. Fix index $k$. 
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for $k \in \mathbb{N}$. Fix index $k$. Now for $x \in E$, we have $|f_{k+1}(x) - f_k(x)| \geq \varepsilon_k$ if and only if $|f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p$, so by Chebychev’s Inequality (page 80) we have
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$$
\|f_{k+1} - f_k\|_p \leq \varepsilon_k^2 \quad \text{and so}
$$

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Theorem 7.6 (continued 1)

Proof (continued).

\[ m(\{ x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k \}) = m(\{ x \in E \mid |f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p \}) \]

\[ \leq \frac{1}{\varepsilon_k^p} \int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^p \text{ by } (\ast). \]

Since \( p \geq 1 \), the series \( \sum_{k=1}^{\infty} \varepsilon_k^p \) converges (\( \varepsilon_k \rightarrow 0 \), so eventually \( \varepsilon_k^p < \varepsilon_k \) and by the Comparison Test). By the Borel-Cantelli Lemma (page 46), since \( \sum_{k=1}^{\infty} m(\{ x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k \}) \leq \sum_{k=1}^{\infty} \varepsilon_k^p < \infty \), almost all \( x \in E \) belong to finitely many of the sets on the left hand side. That is, there is set \( E_0 \subset E \) where \( m(E_0) = 0 \), and for \( x \in E \setminus E_0 \) we have \( x \) in finitely many of the sets on the left hand side.
Theorem 7.6 (continued 1)

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\[ m(\{x \in E \mid |f_{k+1}(x) - f_k(x)| \geq \varepsilon_k\}) = m(\{x \in E \mid |f_{k+1}(x) - f_k(x)|^p \geq \varepsilon_k^p\}) \]

\[ \leq \frac{1}{\varepsilon_k^p} \int_E |f_{k+1} - f_k|^p \leq \varepsilon_k^p \text{ by } (*) . \]

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Proof (continued).

\[ m(\{x \in E \mid \left| f_{k+1}(x) - f_k(x) \right| \geq \varepsilon_k \}) = m(\{x \in E \mid \left| f_{k+1}(x) - f_k(x) \right|^p \geq \varepsilon_k^p \}) \leq \frac{1}{\varepsilon_k^p} \int_E \left| f_{k+1} - f_k \right|^p \leq \varepsilon_k^p \text{ by } (*) . \]

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Theorem 7.6 (continued 2)

Proof (continued). Then for all \( n \geq K(x) \) and all \( k \in \mathbb{N} \) we have

\[
|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)| \leq \sum_{j=n}^{\infty} \varepsilon_j.
\]

Since \( \sum_{j=1}^{\infty} \varepsilon_j \) converges, for \( n \) sufficiently large, the right hand side of this inequality can be made small, and so the sequence of real numbers (for fixed \( x \)) \( \{f_k(x)\} \) is Cauchy and therefore convergent. Define \( f(x) \) as \( \lim_{n \to \infty} f_n(x) = f(x) \). It follows as in the proof of Theorem 7.5 that \( \|f_{n+k} - f_n\|_p \leq \sum_{j=n}^{\infty} \varepsilon_j^2 \) or \( \int_E |f_{n+k} - f_n|^p \leq \left( \sum_{j=n}^{\infty} \varepsilon_j^2 \right)^p \) for all \( n, k \in \mathbb{N} \).

When \( k \to \infty \), \( f_{n+k} \to f \) a.e. on \( E \) and so, by Fatou’s Lemma,

\[
\int_E \lim_{k \to \infty} |f_{n+k} - f_n|^p = \int_E |f - f_n|^p \leq \lim_{k \to \infty} \int |f_{n+k} - f_n|^p \leq \left( \sum_{j=n}^{\infty} \varepsilon_j^2 \right)^p
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for all \( n \in \mathbb{N} \).
Theorem 7.6 (continued 2)

**Proof (continued).** Then for all $n \geq K(x)$ and all $k \in \mathbb{N}$ we have

$$|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)| \leq \sum_{j=n}^{\infty} \varepsilon_j.$$

Since $\sum_{j=1}^{\infty} \varepsilon_j$ converges, for $n$ sufficiently large, the right hand side of this inequality can be made small, and so the sequence of real numbers (for fixed $x$) $\{f_k(x)\}$ is Cauchy and therefore convergent. Define $f(x)$ as $\lim_{n \to \infty} f_n(x) = f(x)$. It follows as in the proof of Theorem 7.5 that

$$\|f_{n+k} - f_n\|_p \leq \sum_{j=n}^{\infty} \varepsilon_j^2$$

or

$$\int_E |f_{n+k} - f_n|^p \leq \left(\sum_{j=n}^{\infty} \varepsilon_j^2\right)^p$$

for all $n, k \in \mathbb{N}$. When $k \to \infty$, $f_{n+k} \to f$ a.e. on $E$ and so, by Fatou’s Lemma,

$$\int_E \lim_{k \to \infty} |f_{n+k} - f_n|^p = \int_E |f - f_n|^p \leq \lim_{k \to \infty} \int |f_{n+k} - f_n|^p \leq \left(\sum_{j=n}^{\infty} \varepsilon_j^2\right)^p$$

for all $n \in \mathbb{N}$. Therefore $f - f_n \in L^p(E)$ and since $f_n \in L^p(E)$, then $f \in L^p(E)$. The right hand side of this inequality can be made arbitrarily small by making $n$ sufficiently large, and so $\{f_n\} \to f$ in $L^p(E)$. So $\{f_n\}$ converges to $f$ in $L^p$ and a.e. pointwise by the construction of $f$. \qed
Theorem 7.6 (continued 2)

Proof (continued). Then for all \( n \geq K(x) \) and all \( k \in \mathbb{N} \) we have

\[
|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)| \leq \sum_{j=n}^{\infty} \varepsilon_j.
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Since \( \sum_{j=1}^{\infty} \varepsilon_j \) converges, for \( n \) sufficiently large, the right hand side of this inequality can be made small, and so the sequence of real numbers (for fixed \( x \)) \( \{f_k(x)\} \) is Cauchy and therefore convergent. Define \( f(x) \) as

\[
\lim_{n \to \infty} f_n(x) = f(x).
\]

It follows as in the proof of Theorem 7.5 that

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for all \( n, k \in \mathbb{N} \).

When \( k \to \infty \), \( f_{n+k} \to f \) a.e. on \( E \) and so, by Fatou’s Lemma,

\[
\int_E \lim_{k \to \infty} |f_{n+k} - f_n|^p = \int_E |f - f_n|^p \leq \lim_{k \to \infty} \int |f_{n+k} - f_n|^p \leq \left( \sum_{j=n}^{\infty} \varepsilon_j^2 \right)^p
\]

for all \( n \in \mathbb{N} \). Therefore \( f - f_n \in L^p(E) \) and since \( f_n \in L^p(E) \), then \( f \in L^p(E) \). The right hand side of this inequality can be made arbitrarily small by making \( n \) sufficiently large, and so \( \{f_n\} \to f \) in \( L^p(E) \). So \( \{f_n\} \) converges to \( f \) in \( L^p \) and a.e. pointwise by the construction of \( f \). \( \square \)
The Riesz-Fischer Theorem. Let $E$ be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p$ then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on $E$ to $f$.

**Proof.** We need to show completeness. Suppose $\{f_n\}$ is a Cauchy sequence in $L^p$. By Proposition 7.5, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that is rapidly Cauchy.
The Riesz-Fischer Theorem. Let $E$ be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p$ then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on $E$ to $f$.

Proof. We need to show completeness. Suppose $\{f_n\}$ is a Cauchy sequence in $L^p$. By Proposition 7.5, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that is rapidly Cauchy. By Theorem 7.6, $\{f_{n_k}\}$ converges to an $f \in L^p(E)$ both with respect to the $L^p$ norm and a.e. pointwise on $E$. By Proposition 7.4, $\{f_n\}$ converges to $f$ with respect to the $L^p$ norm.
The Riesz-Fischer Theorem. Let $E$ be measurable and $1 \leq p \leq \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p$ then there is a subsequence of $\{f_n\}$ which converges pointwise a.e. on $E$ to $f$.

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Theorem 7.7. Let $E$ be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on $E$ to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the $L^p$ norm if and only if

$$\lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

**Proof.** Without loss of generality, we assume $f$ and each $f_n$ is real-valued and the convergence is pointwise on $E$. By Minkowski’s Inequality,

$$\|f_n\|_p = \|f_n - f + f\|_p \leq \|f_n - f\|_p + \|f\|_p,$$

or $\|f_n\|_p - \|f\|_p \leq \|f_n - f\|_p$. Also $\|f\|_p = \|f - f_n + f_n\|_p \leq \|f_n\|_p + \|f - f_n\|_p$ or $-\|f - f_n\|_p \leq \|f_n\|_p - \|f\|_p$. Therefore $\|f_n\|_p - \|f\|_p \leq \|f - f_n\|_p$. So if $\{f_n\} \to f$ with respect to the $L^p$ norm, then $\|f_n\|_p \to \|f\|_p$. To prove the converse, suppose $\|f_n\|_p \to \|f\|_p$ and $\{f_n\} \to f$ pointwise on $E$. 
Theorem 7.7

**Theorem 7.7.** Let $E$ be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on $E$ to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the $L^p$ norm if and only if

$$\lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

**Proof.** Without loss of generality, we assume $f$ and each $f_n$ is real-valued and the convergence is pointwise on $E$. By Minkowski’s Inequality,

$$\|f_n\|_p = \|f_n - f + f\|_p \leq \|f_n - f\|_p + \|f\|_p,$$ or $\|f_n\|_p - \|f\|_p \leq \|f_n - f\|_p$.

Also $\|f\|_p = \|f - f_n + f_n\|_p \leq \|f_n\|_p + \|f - f_n\|_p$ or $-\|f - f_n\|_p \leq \|f_n\|_p - \|f\|_p$. Therefore $\|f_n\|_p - \|f\|_p \leq \|f - f_n\|_p$. So if $\{f_n\} \to f$ with respect to the $L^p$ norm, then $\|f_n\|_p \to \|f\|_p$. To prove the converse, suppose $\|f_n\|_p \to \|f\|_p$ and $\{f_n\} \to f$ pointwise on $E$. 


Theorem 7.7 (continued 1)

Proof (continued). Define $\psi(t) = |t|^p$. Then $\psi$ is “convex” (i.e., concave up since $p \geq 1$) and $\psi((a + b)/2) \leq (\psi(a) + \psi(b))/2$ for all $a, b$. Hence $0 \leq (|a|^p + |b|^p)/2 - |(a - b)/2|^p$ for all $a, b$ (here, we are using $\psi((a + (-b))/2) \leq (\psi(a) + \psi(-b))/2$). Therefore, for each $n$, $h_n$ is nonnegative and measurable on $E$ where $h_n(x) = (|f_n(x)|^p + |f(x)|^p)/2 - |f_n(x) - f(x)|^p/2^p$. Then $h_n \to |f|^p$ since $f_n \to f$ pointwise.
Theorem 7.7 (continued 1)

Proof (continued). Define $\psi(t) = |t|^p$. Then $\psi$ is “convex” (i.e., concave up since $p \geq 1$) and $\psi((a + b)/2) \leq (\psi(a) + \psi(b))/2$ for all $a, b$. Hence $0 \leq (|a|^p + |b|^p)/2 - |(a - b)/2|^p$ for all $a, b$ (here, we are using $\psi((a + (-b))/2) \leq (\psi(a) + \psi(-b))/2$). Therefore, for each $n$, $h_n$ is nonnegative and measurable on $E$ where

$$h_n(x) = (|f_n(x)|^p + |f(x)|^p)/2 - |f_n(x) - f(x)|^p/2^p.$$  

Then $h_n \rightarrow |f|^p$ since $f_n \rightarrow f$ pointwise. So by Fatou’s Lemma and since $\|f_n\|_p \rightarrow \|f\|_p$,

$$\int_E |f|^p \leq \lim inf \int_E h_n = \lim inf \int_E \left(\frac{|f_n|^p + |f|^p}{2} - \frac{|f_n - f|^p}{2^p}\right)$$

$$= \lim inf \left(\int_E \frac{|f_n|^p}{2}\right) + \int_E \frac{|f|^p}{2} - \lim sup \left(\int_E \frac{|f_n - f|^p}{2^p}\right)$$

$$= \int_E \frac{|f|^p}{2} + \int_E \frac{|f|^p}{2} - \lim sup \int_E \frac{|f_n - f|^p}{2^p} = \int_E |f|^p - \lim sup \int_E \frac{|f_n - f|^p}{2^p}.$$
Theorem 7.7 (continued 1)

Proof (continued). Define $\psi(t) = |t|^p$. Then $\psi$ is “convex” (i.e., concave up since $p \geq 1$) and $\psi((a + b)/2) \leq (\psi(a) + \psi(b))/2$ for all $a, b$. Hence $0 \leq (|a|^p + |b|^p)/2 - |(a - b)/2|^p$ for all $a, b$ (here, we are using $\psi((a + (-b))/2) \leq (\psi(a) + \psi(-b))/2$). Therefore, for each $n$, $h_n$ is nonnegative and measurable on $E$ where

$$h_n(x) = (|f_n(x)|^p + |f(x)|^p)/2 - |f_n(x) - f(x)|^p/2^p.$$

Then $h_n \to |f|^p$ since $f_n \to f$ pointwise. So by Fatou’s Lemma and since $\|f_n\|_p \to \|f\|_p$,

$$\int_E |f|^p \leq \liminf \int_E h_n = \liminf \int_E \left( \frac{|f_n|^p + |f|^p}{2} - \frac{|f_n - f|^p}{2^p} \right)$$

$$= \liminf \left( \int_E \frac{|f_n|^p}{2} \right) + \int_E \frac{|f|^p}{2} - \limsup \left( \int_E \frac{|f_n - f|^p}{2^p} \right)$$

$$= \int_E \frac{|f|^p}{2} + \int_E \frac{|f|^p}{2} - \limsup \int_E \frac{|f_n - f|^p}{2^p} = \int_E |f|^p - \limsup \int_E \frac{|f_n - f|^p}{2^p}.$$
Theorem 7.7. Let $E$ be measurable and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on $E$ to $f \in L^p(E)$. Then $\{f_n\} \to f$ with respect to the $L^p$ norm if and only if

$$\lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p,$$

that is $\|f_n\|_p \to \|f\|_p$.

Proof (continued). Therefore $\limsup \int_E |(f_n - f)/2|^p \leq 0$, or $\limsup \int_E |f_n - f|^p \leq 0$ or $\lim \int_E |f_n - f|^p = 0$ (since $|f_n - f|$ nonnegative) and therefore $\|f_n - f\|_p \to 0$, or $\{f_n\} \to f$ with respect to the $L^p$ norm. \qed