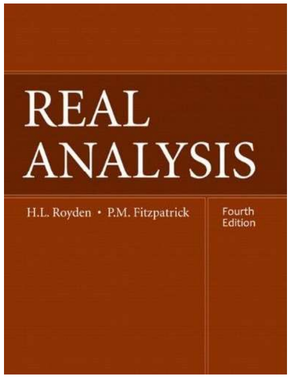
Real Analysis

Chapter 7. The L^p Spaces: Completeness and Approximation

7.4. Approximation and Separability—Proofs of Theorems



Proposition 7.9

Proposition 7.9. Let E be measurable and $1 \le p \le \infty$. Then the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proof. Let $g \in L^p(E)$. Suppose $p = \infty$. Then g is bounded on $E \setminus E_0$ where $m(E_0) = 0$. We infer from the Simple Approximation Lemma (see page 61; really, it is Problem 3.12) that there is a sequence of simple functions on $E \setminus E_0$ that converges uniformly on $E \setminus E_0$ to g, and therefore converges with respect to the L^∞ norm. So simple functions are dense in $L^\infty(E)$.

Now suppose $1 \le p < \infty$. Since g is measurable, by the Simple Approximation Theorem, there is a sequence $\{\varphi_n\}$ of simple functions on E such that $\{\varphi_n\} \to g$ pointwise on E and $|\varphi_n| \le |g|$ on E for all $n \in \mathbb{N}$. By the Integral Comparison Test, each $\varphi_n \in L^p(E)$.

Proposition 7.9 (continued)

Proposition 7.9. Let E be measurable and $1 \le p \le \infty$. Then the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.

Proof (continued). Since $\varphi(t) = t^p$ is convex for $p \ge 1$, then for all $n \in \mathbb{N}$ we have $|\varphi_n - g|^p \le 2^p\{|\varphi_n|^p + |g|^p\} \le 2^{p+1}|g|^p$ on E. Since $|g|^p$ is integrable over E (i.e., $\int_E |g|^p < \infty$), by the Lebesgue Dominated Convergence Theorem

$$\|\varphi_n - g\|_p^p = \int_E |\varphi_n - g|^p \to \int_E |g - g|^p = 0$$

and so $\{\varphi_n\} \to g$ with respect to the L^p norm and simple functions are dense in $L^p(E)$ (by Exercise 7.36, say). □

Proposition 7.10

Proposition 7.10. Let $[a, b]$ be a closed, bounded interval and $1 \le p < \infty$. Then the subspace of step functions on $[a, b]$ is dense in $L^p([a, b])$.

Proof. By Proposition 7.9, simple functions are dense in $L^p([a, b])$, so we only need to show that step functions are dense in the simple functions with respect to the L^p norm. Each simple function is a linear combination of characteristic functions on measurable sets. Therefore, if each such characteristic function can be arbitrarily closely approximated by a step function with respect to the L^p norm then, since a linear combination of step functions are step functions, any simple function can also be approximated arbitrarily closely with respect to the L^p norm by a step function.

Proposition 7.10 (cont.)

Proposition 7.10. Let $[a, b]$ be a closed, bounded interval and $1 \le p < \infty$. Then the subspace of step functions on $[a, b]$ is dense in $L^p([a, b])$.

Proof (continued). Let $g = \chi_A$ where $A \subset [a, b]$ is measurable and let $\varepsilon > 0$. By Theorem 2.12, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which, with $\mathcal{U} = \cup_{k=1}^n I_k$, the symmetric difference $A \triangle \mathcal{U} = (A \setminus \mathcal{U}) \cup (\mathcal{U} \setminus A)$ satisfies $m(A \triangle \mathcal{U}) < \varepsilon^p$. Since $\mathcal{U}$ is a finite disjoint union of open intervals, then $\chi_{\mathcal{U}}$ is a step function. Moreover,

$$\|\chi_A - \chi_{\mathcal{U}}\|_p = \left\{ \int_{[a,b]} |\chi_A - \chi_{\mathcal{U}}|^p \right\}^{1/p} = \left\{ \int_{[a,b]} |\chi_A - \chi_{\mathcal{U}}| \right\}^{1/p}$$

$$\le \left\{ \int_{A \triangle \mathcal{U}} 1 \right\}^{1/p} = (m(A \triangle \mathcal{U}))^{1/p} < (\varepsilon^p)^{1/p} = \varepsilon.$$

So step function $\chi_{\mathcal{U}}$ approximates characteristic function χ_A to within $\varepsilon > 0$ with respect to the L^p norm, and the result follows. □

Theorem 7.11

Theorem 7.11. Let E be measurable and $1 \le p < \infty$. Then $L^p(E)$ is separable.

Proof. Consider $S([a, b])$, the set of step functions on $[a, b]$. Define $S'([a, b])$ to the subset of $S([a, b])$ which consists of step functions ψ on $[a, b]$ that take on rational values and for which there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with ψ constant on (x_{k-1}, x_k) for $1 \le k \le n$ and x_k rational for $1 \le k \le n - 1$ ($x_0 = a$ and $x_n = b$ "always"). Then $S'([a, b])$ is countable and $S'([a, b])$ is dense in $S([a, b])$ with respect to the L^p norm by Exercise 7.40. By Proposition 7.10, $S([a, b])$ is dense in $L^p([a, b])$ and so $S'([a, b])$ is dense in $L^p([a, b])$ (we have $S'([a, b]) \subset S([a, b]) \subset L^p([a, b])$). For each $n \in \mathbb{N}$, define $\mathcal{F}_n$ to be the functions on $\mathbb{R}$ that vanish outside $[-n, n]$ and whose restrictions to $[-n, n]$ belong to $S'([-n, n])$. Define $\mathcal{F} = \cup_{n \in \mathbb{N}} \mathcal{F}_n$. Then $\mathcal{F}$ is countable and $\mathcal{F} \subset L^p(\mathbb{R})$. For any $f \in L^p(E)$, define $f_n = f\chi_{[-n,n]}$ and notice that $\lim_{n \to \infty} f_n = f$ (pointwise).

Theorem 7.11 (cont.)

Theorem 7.11. Let E be measurable and $1 \le p < \infty$. Then $L^p(E)$ is separable.

Proof (continued). By the Monotone Convergence Theorem,

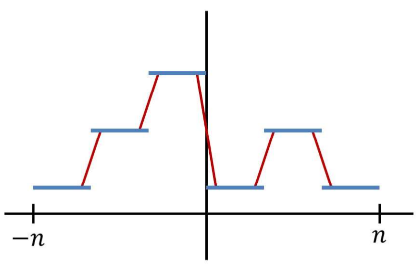
$$\lim_{n \to \infty} \int_{\mathbb{R}} |f_n|^p = \int_{\mathbb{R}} |f|^p.$$

Therefore by Theorem 7.7, $\{f_n\} \to f$ with respect to the $L^p(\mathbb{R})$ norm. So $\mathcal{F}$ is dense in $L^p(\mathbb{R})$ with respect to the $L^p(\mathbb{R})$ norm by Exercise 7.36. Finally, for any measurable set E, replace $\mathcal{F}$ and each $\mathcal{F}_n$ with functions restricted to E to get that $\mathcal{F}$ (restricted to E) is dense in $L^p(E)$. Therefore $L^p(E)$ is separable. □

Theorem 7.12

Theorem 7.12. Let E be measurable and $1 \le p < \infty$. Then $C_c(E)$, the linear space of continuous functions on E that vanish outside a bounded set, is dense in $L^p(E)$.

Idea of Proof. By Theorem 7.11, we know that $S'([a, b])$ is dense in $L^p([a, b])$. The idea is to approximate each element of $\mathcal{F}$ with a continuous function. We do so as follows



This, combined with restrictions of functions to E, proves the result. □