Real Analysis

Chapter 7. The L^p Spaces: Completeness and Approximation

7.4. Approximation and Separability—Proofs of Theorems

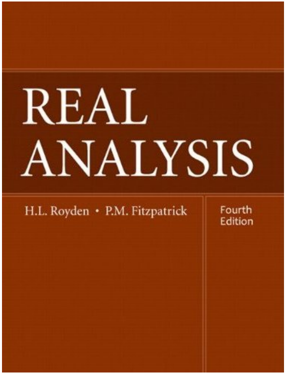

Table of contents

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Now suppose $1 \leq p < \infty$. Since g is measurable, by the Simple Approximation Theorem, there is a sequence $\{\varphi_n\}$ of simple functions on E such that $\{\varphi_n\} \to g$ pointwise on E and $|\varphi_n| \leq |g|$ on E for all $n \in \mathbb{N}$. By the Integral Comparison Test, each $\varphi_n \in L^p(E)$.

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Proof (continued). Since $\varphi(t) = t^p$ is convex for $p \geq 1$, then for all $n \in \mathbb{N}$ we have $|\varphi_n - g|^p \leq 2^p\{|\varphi_n|^p + |g|^p\} \leq 2^{p+1}|g|^p$ on E. Since $|g|^p$ is integrable over E (i.e., $\int_E |g|^p < \infty$), by the Lebesgue Dominated Convergence Theorem

$$\|\varphi_n - g\|_p^p = \int_E |\varphi_n - g|^p \to \int_E |g - g|^p = 0$$

and so $\{\varphi_n\} \to g$ with respect to the L^p norm and simple functions are dense in $L^p(E)$ (by Exercise 7.36, say). □

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Proposition 7.10

Proposition 7.10. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then the subspace of step functions on $[a, b]$ is dense in $L^p([a, b])$.

Proof. By Proposition 7.9, simple functions are dense in $L^p([a, b])$, so we only need to show that step functions are dense in the simple functions with respect to the L^p norm. Each simple function is a linear combination of characteristic functions on measurable sets.

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Proof (continued). Let $g = \chi_A$ where $A \subset [a, b]$ is measurable and let $\varepsilon > 0$. By Theorem 2.12, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which, with $\mathcal{U} = \dot\cup_{k=1}^n I_k$, the symmetric difference $A \triangle \mathcal{U} = (A \setminus \mathcal{U}) \cup (\mathcal{U} \setminus A)$ satisfies $m(A \triangle \mathcal{U}) < \varepsilon^p$. Since $\mathcal{U}$ is a finite disjoint union of open intervals, then $\chi_{\mathcal{U}}$ is a step function. Moreover,

$$\|\chi_A - \chi_{\mathcal{U}}\|_p = \left\{\int_{[a,b]} |\chi_A - \chi_{\mathcal{U}}|^p\right\}^{1/p} = \left\{\int_{[a,b]} |\chi_A - \chi_{\mathcal{U}}|\right\}^{1/p}$$

$$\leq \left\{\int_{A \triangle \mathcal{U}} 1\right\}^{1/p} = (m(A \triangle \mathcal{U}))^{1/p} < (\varepsilon^p)^{1/p} = \varepsilon.$$

So step function $\chi_{\mathcal{U}}$ approximates characteristic function χ_A to within $\varepsilon > 0$ with respect to the L^p norm, and the result follows. □

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Theorem 7.11

Theorem 7.11. Let E be measurable and $1 \leq p < \infty$. Then $L^p(E)$ is separable.

Proof. Consider $S([a,b])$, the set of step functions on $[a,b]$. Define $S'([a,b])$ to the subset of $S([a,b])$ which consists of step functions ψ on $[a,b]$ that take on rational values and for which there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with ψ constant on (x_{k-1}, x_k) for $1 \leq k \leq n$ and x_k rational for $1 \leq k \leq n-1$ ($x_0 = a$ and $x_n = b$ "always").

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Proof (continued). By the Monotone Convergence Theorem,

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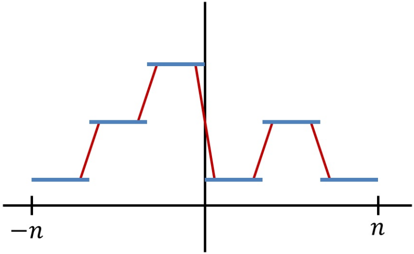

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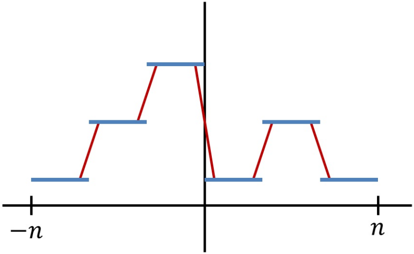

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