Real Analysis

Chapter 8. The L^p Spaces: Duality and Weak Convergence

8.1. The Riesz Representation for the Dual of L^p, $1 \leq 1 < \infty$—Proofs of Theorems

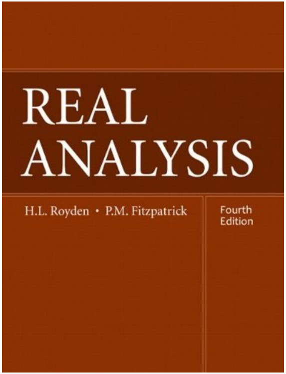

Table of contents

Proposition 8.2

Proposition 8.2. Let E be measurable, $1 \leq p < \infty$, q the conjugate of p, and g belong to $L^q(E)$. Define the functional T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Proof. First, for $f_1, f_2 \in L^p(E)$ and $\alpha, \beta \in \mathbb{R}$ we have $T(\alpha f_1 + \beta f_2) = \int_E g(\alpha f_1 + \beta f_2) = \alpha \int_E gf_1 + \beta \int_E gf_2 = \alpha T(f_1) + \beta T(f_2)$, and so T is linear. Since $|T(f)| \leq \|g\|_q \|f\|_p$ by Hölder's Inequality, we see that T is a bounded linear functional on $L^p(E)$ and $\|T\|_* \leq \|g\|_q$.

Proposition 8.2

Proposition 8.2. Let E be measurable, $1 \leq p < \infty$, q the conjugate of p, and g belong to $L^q(E)$. Define the functional T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Proof. First, for $f_1, f_2 \in L^p(E)$ and $\alpha, \beta \in \mathbb{R}$ we have $T(\alpha f_1 + \beta f_2) = \int_E g(\alpha f_1 + \beta f_2) = \alpha \int_E gf_1 + \beta \int_E gf_2 = \alpha T(f_1) + \beta T(f_2)$, and so T is linear. Since $|T(f)| \leq \|g\|_q \|f\|_p$ by Hölder's Inequality, we see that T is a bounded linear functional on $L^p(E)$ and $\|T\|_* \leq \|g\|_q$. By Theorem 7.1 (with p and q interchanged) the conjugate of $g \in L^q$ is

$$g^* = \|g\|_q^{1-q} \text{sgn}(g)|g|^{q-1} \in L^p$$

and $T(g^*) = \int_E gg^* = \|g\|_q$ where $\|g^*\|_p = 1$. Therefore $\|T\|_* = \|g\|_q$. □

Proposition 8.2

Proposition 8.2. Let E be measurable, $1 \leq p < \infty$, q the conjugate of p, and g belong to $L^q(E)$. Define the functional T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Proof. First, for $f_1, f_2 \in L^p(E)$ and $\alpha, \beta \in \mathbb{R}$ we have $T(\alpha f_1 + \beta f_2) = \int_E g(\alpha f_1 + \beta f_2) = \alpha \int_E gf_1 + \beta \int_E gf_2 = \alpha T(f_1) + \beta T(f_2)$, and so T is linear. Since $|T(f)| \leq \|g\|_q \|f\|_p$ by Hölder's Inequality, we see that T is a bounded linear functional on $L^p(E)$ and $\|T\|_* \leq \|g\|_q$. By Theorem 7.1 (with p and q interchanged) the conjugate of $g \in L^q$ is

$$g^* = \|g\|_q^{1-q} \text{sgn}(g)|g|^{q-1} \in L^p$$

and $T(g^*) = \int_E gg^* = \|g\|_q$ where $\|g^*\|_p = 1$. Therefore $\|T\|_* = \|g\|_q$. □

Proposition 8.3

Proposition 8.3. Let T and S be bounded linear functionals on a normed linear space X. If $T = S$ on a dense subset X_0 of X, then $T = S$ on X.

Proof. Let $g \in X$. Since X_0 is dense in X, then by Note 7.4.A (or Exercise 7.36) there is a sequence $\{g_n\} \subset X_0$ such that $g_n \to g$. From Note 8.1.A, we have $S(g_n) \to S(g)$ and $T(g_n) \to T(g)$. Since $S(g_n) = T(g_n)$ for all $n \in \mathbb{N}$, then $S(g) = T(g)$. □

Proposition 8.3

Proposition 8.3. Let T and S be bounded linear functionals on a normed linear space X. If $T = S$ on a dense subset X_0 of X, then $T = S$ on X.

Proof. Let $g \in X$. Since X_0 is dense in X, then by Note 7.4.A (or Exercise 7.36) there is a sequence $\{g_n\} \subset X_0$ such that $g_n \to g$. From Note 8.1.A, we have $S(g_n) \to S(g)$ and $T(g_n) \to T(g)$. Since $S(g_n) = T(g_n)$ for all $n \in \mathbb{N}$, then $S(g) = T(g)$. □

Lemma 8.4

Lemma 8.4. Let E be measurable and $1 \leq p < \infty$. Suppose g is integrable over E and there is $M > 0$ such that $|\int_E gf| \leq M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where q is the conjugate of p. Moreover, $\|g\|_q \leq M$.

Proof. Since g is integrable over E, it is finite a.e. on E by Proposition 4.15. So Without loss of generality (or by excising a set of measure zero from E), we assume g is finite on all of E.

Lemma 8.4

Lemma 8.4. Let E be measurable and $1 \le p < \infty$. Suppose g is integrable over E and there is $M > 0$ such that $|\int_E gf| \le M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where q is the conjugate of p. Moreover, $\|g\|_q \le M$.

Proof. Since g is integrable over E, it is finite a.e. on E by Proposition 4.15. So Without loss of generality (or by excising a set of measure zero from E), we assume g is finite on all of E. We first consider the case $p > 1$. Since $|g|$ is a nonnegative measurable function, by the Simple Approximation Theorem, there is a sequence of simple functions $\{\varphi_n\}$ that converges pointwise on E to $|g|$ and $0 \le \varphi_n \le |g|$ on E for all $n \in \mathbb{N}$. So $\{\varphi_n^q\}$ is a sequence of nonnegative measurable functions that converges pointwise on E to $|g|^q$, and by Fatou's Lemma $\int_E |g|^q \le \liminf \int_E \varphi_n^q$.

Lemma 8.4

Lemma 8.4. Let E be measurable and $1 \leq p < \infty$. Suppose g is integrable over E and there is $M > 0$ such that $|\int_E gf| \leq M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where q is the conjugate of p. Moreover, $\|g\|_q \leq M$.

Proof. Since g is integrable over E, it is finite a.e. on E by Proposition 4.15. So Without loss of generality (or by excising a set of measure zero from E), we assume g is finite on all of E. We first consider the case $p > 1$. Since $|g|$ is a nonnegative measurable function, by the Simple Approximation Theorem, there is a sequence of simple functions $\{\varphi_n\}$ that converges pointwise on E to $|g|$ and $0 \leq \varphi_n \leq |g|$ on E for all $n \in \mathbb{N}$. So $\{\varphi_n^q\}$ is a sequence of nonnegative measurable functions that converges pointwise on E to $|g|^q$, and by Fatou's Lemma $\int_E |g|^q \leq \liminf \int_E \varphi_n^q$. So to show that $|g|^q$ is integrable over E and $\|g\|_q \leq M$ it suffices to show that

$$\int_E \varphi_n^q \leq M^q \text{ for all } n \in \mathbb{N}. \qquad (10)$$

Lemma 8.4

Lemma 8.4. Let E be measurable and $1 \leq p < \infty$. Suppose g is integrable over E and there is $M > 0$ such that $|\int_E gf| \leq M\|f\|_p$ for every simple function $f \in L^p(E)$. Then $g \in L^q(E)$ where q is the conjugate of p. Moreover, $\|g\|_q \leq M$.

Proof. Since g is integrable over E, it is finite a.e. on E by Proposition 4.15. So Without loss of generality (or by excising a set of measure zero from E), we assume g is finite on all of E. We first consider the case $p > 1$. Since $|g|$ is a nonnegative measurable function, by the Simple Approximation Theorem, there is a sequence of simple functions $\{\varphi_n\}$ that converges pointwise on E to $|g|$ and $0 \leq \varphi_n \leq |g|$ on E for all $n \in \mathbb{N}$. So $\{\varphi_n^q\}$ is a sequence of nonnegative measurable functions that converges pointwise on E to $|g|^q$, and by Fatou's Lemma $\int_E |g|^q \leq \liminf \int_E \varphi_n^q$. So to show that $|g|^q$ is integrable over E and $\|g\|_q \leq M$ it suffices to show that

$$\int_E \varphi_n^q \leq M^q \text{ for all } n \in \mathbb{N}. \qquad (10)$$

Lemma 8.4 (continued 1)

Proof (continued). Let $n \in \mathbb{N}$ be fixed. We have

$$\varphi_n^q = \varphi_n \varphi_n^{q-1} \leq |g| \varphi_n^{q-1} = g\,\mathrm{sgn}(g) \varphi_n^{q-1} \text{ on } E. \quad (11)$$

Define simple function f_n as $f_n = \mathrm{sgn}(g)\varphi_n^{q-1}$ on E. The function φ_n is integrable over E, since it is dominated on E by the integrable function $|g|$, by the Integral Comparison Test (Proposition 4.16). Therefore, since φ_n is simple, then φ_n has finite support and hence f_n has finite support and is bounded, so $f_n \in L^p(E)$.

Lemma 8.4 (continued 1)

Proof (continued). Let $n \in \mathbb{N}$ be fixed. We have

$$\varphi_n^q = \varphi_n \varphi_n^{q-1} \leq |g| \varphi_n^{q-1} = g\text{sgn}(g)\varphi_n^{q-1} \text{ on } E. \quad (11)$$

Define simple function f_n as $f_n = \text{sgn}(g)\varphi_n^{q-1}$ on E. The function φ_n is integrable over E, since it is dominated on E by the integrable function $|g|$, by the Integral Comparison Test (Proposition 4.16). Therefore, since φ_n is simple, then φ_n has finite support and hence f_n has finite support and is bounded, so $f_n \in L^p(E)$. We now have

$$\begin{aligned} \int_E \varphi_n^q &\leq \int_E g\text{sgn}(g)\varphi_n^{q-1} \text{ by monotonicity and (11)} \\ &= \int_E g f_n \text{ by definition of } f_n \\ &\leq M\|f_n\|_p \text{ by hypothesis.} \quad (12) \end{aligned}$$

Lemma 8.4 (continued 1)

Proof (continued). Let $n \in \mathbb{N}$ be fixed. We have

$$\varphi_n^q = \varphi_n \varphi_n^{q-1} \leq |g|\varphi_n^{q-1} = g\,\text{sgn}(g)\varphi_n^{q-1} \text{ on } E. \quad (11)$$

Define simple function f_n as $f_n = \text{sgn}(g)\varphi_n^{q-1}$ on E. The function φ_n is integrable over E, since it is dominated on E by the integrable function $|g|$, by the Integral Comparison Test (Proposition 4.16). Therefore, since φ_n is simple, then φ_n has finite support and hence f_n has finite support and is bounded, so $f_n \in L^p(E)$. We now have

$$\begin{aligned}
\int_E \varphi_n^q &\leq \int_E g\,\text{sgn}(g)\varphi_n^{q-1} \text{ by monotonicity and (11)} \\
&= \int_E g f_n \text{ by definition of } f_n \\
&\leq M\|f_n\|_p \text{ by hypothesis.} \quad (12)
\end{aligned}$$

Lemma 8.4 (continued 2)

Proof (continued). Since q is the conjugate of p, then $p(q-1) = q$ and so $\int_E |f_n|^p = \int_E \varphi_n^{p(q-1)} = \int_E \varphi_n^q$. We rewrite (12) as $\int_E \varphi_n^q \leq M \left(\int_E |f_n|^p\right)^{1/p} = M \left(\int_E \varphi_n^q\right)^{1/p}$. Since φ_n^q is integrable over E by (12), neither side of this inequality is ∞ and so $\left(\int_E \varphi_n^q\right)^{1-1/p} \leq M$. Since $1 - 1/p = 1/q$, this implies $\|\varphi_n\| \leq M$, which is equivalent to (10) and the result follows for $p > 1$.

Lemma 8.4 (continued 2)

Proof (continued). Since q is the conjugate of p, then $p(q-1) = q$ and so $\int_E |f_n|^p = \int_E \varphi_n^{p(q-1)} = \int_E \varphi_n^q$. We rewrite (12) as $\int_E \varphi_n^q \leq M \left(\int_E |f_n|^p\right)^{1/p} = M \left(\int_E \varphi_n^q\right)^{1/p}$. Since φ_n^q is integrable over E by (12), neither side of this inequality is ∞ and so $\left(\int_E \varphi_n^q\right)^{1-1/p} \leq M$. Since $1 - 1/p = 1/q$, this implies $\|\varphi_n\| \leq M$, which is equivalent to (10) and the result follows for $p > 1$.

Now consider the case $p = 1$. ASSUME integrable function g satisfies the hypotheses but $g \notin L^q(E) = L^\infty(E)$. Then parameter M in the hypotheses is not an essential upper bound for g (i.e., $\|g\|_\infty > M$).

Lemma 8.4 (continued 2)

Proof (continued). Since q is the conjugate of p, then $p(q-1) = q$ and so $\int_E |f_n|^p = \int_E \varphi_n^{p(q-1)} = \int_E \varphi_n^q$. We rewrite (12) as $\int_E \varphi_n^q \leq M \left(\int_E |f_n|^p\right)^{1/p} = M \left(\int_E \varphi_n^q\right)^{1/p}$. Since φ_n^q is integrable over E by (12), neither side of this inequality is ∞ and so $\left(\int_E \varphi_n^q\right)^{1-1/p} \leq M$. Since $1 - 1/p = 1/q$, this implies $\|\varphi_n\| \leq M$, which is equivalent to (10) and the result follows for $p > 1$.

Now consider the case $p = 1$. ASSUME integrable function g satisfies the hypotheses but $g \notin L^q(E) = L^\infty(E)$. Then parameter M in the hypotheses is not an essential upper bound for g (i.e., $\|g\|_\infty > M$). Let $E_{1/n} = \{x \in E \mid |g(x)| > M + 1/n\}$ for $n \in \mathbb{N}$. Then $E_{1/n} \subset E_{1/(n+1)}$ and $\lim_{n\to\infty} E_n = \{x \in E \mid |g(x)| > M\}$. By assumption, $m(\{x \in E \mid |g(x)| > M\} > 0$.

Lemma 8.4 (continued 2)

Proof (continued). Since q is the conjugate of p, then $p(q-1) = q$ and so $\int_E |f_n|^p = \int_E \varphi_n^{p(q-1)} = \int_E \varphi_n^q$. We rewrite (12) as $\int_E \varphi_n^q \le M \left(\int_E |f_n|^p \right)^{1/p} = M \left(\int_E \varphi_n^q \right)^{1/p}$. Since φ_n^q is integrable over E by (12), neither side of this inequality is ∞ and so $\left(\int_E \varphi_n^q \right)^{1-1/p} \le M$. Since $1 - 1/p = 1/q$, this implies $\|\varphi_n\| \le M$, which is equivalent to (10) and the result follows for $p > 1$.

Now consider the case $p = 1$. ASSUME integrable function g satisfies the hypotheses but $g \notin L^q(E) = L^\infty(E)$. Then parameter M in the hypotheses is not an essential upper bound for g (i.e., $\|g\|_\infty > M$). Let $E_{1/n} = \{x \in E \mid |g(x)| > M + 1/n\}$ for $n \in \mathbb{N}$. Then $E_{1/n} \subset E_{1/(n+1)}$ and $\lim_{n\to\infty} E_n = \{x \in E \mid |g(x)| > M\}$. By assumption, $m(\{x \in E \mid |g(x)| > M\} > 0$. By Continuity of Measure (Theorem 2.15), $\lim_{n\to\infty} m(E_{1/n}) = m(\{x \in E \mid |g(x)| > M\}) > 0$. So $m(E_{1/N}) > 0$ for some $N \in \mathbb{N}$. Let E' be a measurable subset of $E_{1/N}$ such that E' has finite positive measure.

Lemma 8.4 (continued 2)

Proof (continued). Since q is the conjugate of p, then $p(q-1) = q$ and so $\int_E |f_n|^p = \int_E \varphi_n^{p(q-1)} = \int_E \varphi_n^q$. We rewrite (12) as $\int_E \varphi_n^q \le M \left(\int_E |f_n|^p\right)^{1/p} = M \left(\int_E \varphi_n^q\right)^{1/p}$. Since φ_n^q is integrable over E by (12), neither side of this inequality is ∞ and so $\left(\int_E \varphi_n^q\right)^{1-1/p} \le M$. Since $1 - 1/p = 1/q$, this implies $\|\varphi_n\| \le M$, which is equivalent to (10) and the result follows for $p > 1$.

Now consider the case $p = 1$. ASSUME integrable function g satisfies the hypotheses but $g \notin L^q(E) = L^\infty(E)$. Then parameter M in the hypotheses is not an essential upper bound for g (i.e., $\|g\|_\infty > M$). Let $E_{1/n} = \{x \in E \mid |g(x)| > M + 1/n\}$ for $n \in \mathbb{N}$. Then $E_{1/n} \subset E_{1/(n+1)}$ and $\lim_{n\to\infty} E_n = \{x \in E \mid |g(x)| > M\}$. By assumption, $m(\{x \in E \mid |g(x)| > M\} > 0$. By Continuity of Measure (Theorem 2.15), $\lim_{n\to\infty} m(E_{1/n}) = m(\{x \in E \mid |g(x)| > M\}) > 0$. So $m(E_{1/N}) > 0$ for some $N \in \mathbb{N}$. Let E' be a measurable subset of $E_{1/N}$ such that E' has finite positive measure.

Lemma 8.4 (continued 3)

Proof (continued). Define $f = \text{sgn}(g)\chi_{E'}$. Then f is simple and $f \in L^1(E)$ with $\|f\|_1 = \left|\int_E \text{sgn}(g)\chi_{E'}\right| = m(E') > 0$. But,

$$\left|\int_E gf\right| = \left|\int_E g\text{sgn}(g)\chi_{E'}\right| = \left|\int_{E'} g\text{sgn}(g)\right|$$

$$= \int_{E'} |g| \geq (M + 1/N)m(E') > Mm(E') = M\|f\|_1,$$

a CONTRADICTION to the hypotheses of the lemma.

Lemma 8.4 (continued 3)

Proof (continued). Define $f = \text{sgn}(g)\chi_{E'}$. Then f is simple and $f \in L^1(E)$ with $\|f\|_1 = \left|\int_E \text{sgn}(g)\chi_{E'}\right| = m(E') > 0$. But,

$$\left|\int_E gf\right| = \left|\int_E g\text{sgn}(g)\chi_{E'}\right| = \left|\int_{E'} g\text{sgn}(g)\right|$$

$$= \int_{E'} |g| \geq (M + 1/N)m(E') > Mm(E') = M\|f\|_1,$$

a CONTRADICTION to the hypotheses of the lemma. So the assumption that $g \notin L^q(E)$ is false, and in fact $g \in L^q(E)$. In addition $\|g\|_q = \|g\|_\infty \leq M$ (or else the argument above yields the same contradiction). □

Lemma 8.4 (continued 3)

Proof (continued). Define $f = \text{sgn}(g)\chi_{E'}$. Then f is simple and $f \in L^1(E)$ with $\|f\|_1 = \left|\int_E \text{sgn}(g)\chi_{E'}\right| = m(E') > 0$. But,

$$\left|\int_E gf\right| = \left|\int_E g\text{sgn}(g)\chi_{E'}\right| = \left|\int_{E'} g\text{sgn}(g)\right|$$

$$= \int_{E'} |g| \geq (M + 1/N)m(E') > Mm(E') = M\|f\|_1,$$

a CONTRADICTION to the hypotheses of the lemma. So the assumption that $g \notin L^q(E)$ is false, and in fact $g \in L^q(E)$. In addition $\|g\|_q = \|g\|_\infty \leq M$ (or else the argument above yields the same contradiction). □

Theorem 8.5

Theorem 8.5. Let $1 \leq p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof. Let $p > 1$ (the case $p = 1$ is similar). For $x \in [a,b]$, define $\Phi(x) = T(\chi_{[a,x)})$. For each $[c,d) \subset [a,b]$ we have $\chi_{[c,d)} = \chi_{[a,d)} - \chi_{[a,c)}$ and since T is linear

$$\Phi(d) - \Phi(c) = T(\chi_{[a,d)}) - T(\chi_{[a,c)}) = T(\chi_{[a,d)} - \chi_{[a,c)}) = T(\chi_{[c,d)}).$$

Theorem 8.5

Theorem 8.5. Let $1 \le p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof. Let $p > 1$ (the case $p = 1$ is similar). For $x \in [a,b]$, define $\Phi(x) = T(\chi_{[a,x)})$. For each $[c,d) \subset [a,b]$ we have $\chi_{[c,d)} = \chi_{[a,d)} - \chi_{[a,c)}$ and since T is linear

$$\Phi(d) - \Phi(c) = T(\chi_{[a,d)}) - T(\chi_{[a,c)}) = T(\chi_{[a,d)} - \chi_{[a,c)}) = T(\chi_{[c,d)}).$$

So for $\{[a_k,b_k)\}_{k=1}^n$ a finite disjoint collection of intervals in $[a,b]$,

$$\sum_{k=1}^n |\Phi(b_k) - \Phi(a_k)| = \sum_{k=1}^n \varepsilon_k T(\chi_{[a_k,b_k)}) = T\left(\sum_{k=1}^n \varepsilon_k \chi_{[a_k,b_k)}\right)$$

where $\varepsilon_k = \mathrm{sgn}(\Phi(b_k) - \Phi(a_k))$. So for simple function $f = \sum_{k=1}^n \varepsilon_k \chi_{[a_k,b_k)}$, we have $|T(f)| \le \|T\|_* \|f\|_p$ and $\|f\|_p = \left\{\int_{[a,b]} |f|^p\right\}^{1/p} = \left(\sum_{k=1}^n (b_k - a_k)\right)^{1/p}$.

Theorem 8.5

Theorem 8.5. Let $1 \leq p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof. Let $p > 1$ (the case $p = 1$ is similar). For $x \in [a,b]$, define $\Phi(x) = T(\chi_{[a,x)})$. For each $[c,d) \subset [a,b]$ we have $\chi_{[c,d)} = \chi_{[a,d)} - \chi_{[a,c)}$ and since T is linear

$$\Phi(d) - \Phi(c) = T(\chi_{[a,d)}) - T(\chi_{[a,c)}) = T(\chi_{[a,d)} - \chi_{[a,c)}) = T(\chi_{[c,d)}).$$

So for $\{[a_k,b_k)\}_{k=1}^n$ a finite disjoint collection of intervals in $[a,b]$,

$$\sum_{k=1}^n |\Phi(b_k) - \Phi(a_k)| = \sum_{k=1}^n \varepsilon_k T(\chi_{[a_k,b_k)}) = T\left(\sum_{k=1}^n \varepsilon_k \chi_{[a_k,b_k)}\right)$$

where $\varepsilon_k = \text{sgn}(\Phi(b_k) - \Phi(a_k))$. So for simple function $f = \sum_{k=1}^n \varepsilon_k \chi_{[a_k,b_k)}$, we have $|T(f)| \leq \|T\|_* \|f\|_p$ and $\|f\|_p = \left\{\int_{[a,b]} |f|^p\right\}^{1/p} = \left(\sum_{k=1}^n (b_k - a_k)\right)^{1/p}$.

Theorem 8.5 (continued 1)

Proof (continued). So for given $\varepsilon > 0$, with $\delta = (\varepsilon / \| T \|_*)^p$, we have that $\sum_{k=1}^n (b_k - a_k) < \delta$ implies

$$\sum_{k=1}^{n} |\Phi(b_k) - \Phi(a_k)| \leq \| T \|_* \left((\varepsilon / \| T \|_*)^p \right)^{1/p} = \varepsilon.$$

That is, Φ is absolutely continuous on $[a, b]$ (see Section 6.4. Absolutely Continuous Functions for the definition).

By Theorem 6.10 in Section 6.5, $g = \Phi'$ is integrable over $[a, b]$ and $\Phi(x) = \int_a^x g$ for all $x \in [a, b]$. So for each $[c, d] \subset (a, b)$,

$$T(\chi_{[c,d)}) = \Phi(d) - \Phi(c) = \int_a^d g - \int_a^c g = \int_c^d g = \int_a^b g \chi_{[c,d)}.$$

Theorem 8.5 (continued 1)

Proof (continued). So for given $\varepsilon > 0$, with $\delta = (\varepsilon/\|T\|_*)^p$, we have that $\sum_{k=1}^n (b_k - a_k) < \delta$ implies

$$\sum_{k=1}^n |\Phi(b_k) - \Phi(a_k)| \leq \|T\|_* \left((\varepsilon/\|T\|_*)^p\right)^{1/p} = \varepsilon.$$

That is, Φ is absolutely continuous on $[a, b]$ (see Section 6.4. Absolutely Continuous Functions for the definition).

By Theorem 6.10 in Section 6.5, $g = \Phi'$ is integrable over $[a, b]$ and $\Phi(x) = \int_a^x g$ for all $x \in [a, b]$. So for each $[c, d] \subset (a, b)$,

$$T(\chi_{[c,d)}) = \Phi(d) - \Phi(c) = \int_a^d g - \int_a^c g = \int_c^d g = \int_a^b g\chi_{[c,d)}.$$

For *step* function $f = \sum_{k=1}^n y_k \chi_{(a_k,b_k)}$ on $[a, b]$ (we are not concerned about the values of f at the endpoints of the subintervals since T involves integration), we have...

Theorem 8.5 (continued 1)

Proof (continued). So for given $\varepsilon > 0$, with $\delta = (\varepsilon / \|T\|_*)^p$, we have that $\sum_{k=1}^n (b_k - a_k) < \delta$ implies

$$\sum_{k=1}^{n} |\Phi(b_k) - \Phi(a_k)| \leq \|T\|_* \left((\varepsilon / \|T\|_*)^p\right)^{1/p} = \varepsilon.$$

That is, Φ is absolutely continuous on $[a, b]$ (see Section 6.4. Absolutely Continuous Functions for the definition).

By Theorem 6.10 in Section 6.5, $g = \Phi'$ is integrable over $[a, b]$ and $\Phi(x) = \int_a^x g$ for all $x \in [a, b]$. So for each $[c, d] \subset (a, b)$,

$$T(\chi_{[c,d)}) = \Phi(d) - \Phi(c) = \int_a^d g - \int_a^c g = \int_c^d g = \int_a^b g\chi_{[c,d)}.$$

For *step* function $f = \sum_{k=1}^n y_k \chi_{(a_k, b_k)}$ on $[a, b]$ (we are not concerned about the values of f at the endpoints of the subintervals since T involves integration), we have...

Theorem 8.5 (continued 2)

Proof (continued).

$$T(f) = T\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \sum_{k=1}^{n} y_k T(\chi_{(a_k,b_k)}) = \sum_{k=1}^{n} y_k \int_a^b g\chi_{(a_k,b_k)}$$

$$= \int_a^b g\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \int_a^b gf. \qquad (*)$$

By Proposition 7.10, for simple f on $[a,b]$ there is a sequence of step functions $\{\varphi_n\}$ that converges to f with respect to the L^p norm and is uniformly pointwise bounded on $[a,b]$. Since T is linear and bounded on $L^p([a,b])$, then $\lim_{n\to\infty} T(\varphi_n) = T(f)$ by Note 8.1.A.

Theorem 8.5 (continued 2)

Proof (continued).

$$T(f) = T\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \sum_{k=1}^{n} y_k T(\chi_{(a_k,b_k)}) = \sum_{k=1}^{n} y_k \int_a^b g\chi_{(a_k,b_k)}$$

$$= \int_a^b g\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \int_a^b gf. \qquad (*)$$

By Proposition 7.10, for simple f on $[a,b]$ there is a sequence of step functions $\{\varphi_n\}$ that converges to f with respect to the L^p norm and is uniformly pointwise bounded on $[a,b]$. Since T is linear and bounded on $L^p([a,b])$, then $\lim_{n\to\infty} T(\varphi_n) = T(f)$ by Note 8.1.A. By the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty}\left(\int_a^b g\varphi_n\right) = \int_a^b gf$ (g is integrable on $[a,b]$ and φ_n is uniformly pointwise bounded, so this provides the bound on $g\varphi_n$ for the Lebesgue Dominated Convergence Theorem).

Theorem 8.5 (continued 2)

Proof (continued).

$$T(f) = T\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \sum_{k=1}^{n} y_k T(\chi_{(a_k,b_k)}) = \sum_{k=1}^{n} y_k \int_a^b g\chi_{(a_k,b_k)}$$

$$= \int_a^b g\left(\sum_{k=1}^{n} y_k \chi_{(a_k,b_k)}\right) = \int_a^b gf. \qquad (*)$$

By Proposition 7.10, for simple f on $[a,b]$ there is a sequence of step functions $\{\varphi_n\}$ that converges to f with respect to the L^p norm and is uniformly pointwise bounded on $[a,b]$. Since T is linear and bounded on $L^p([a,b])$, then $\lim_{n\to\infty} T(\varphi_n) = T(f)$ by Note 8.1.A. By the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty}\left(\int_a^b g\varphi_n\right) = \int_a^b gf$ (g is integrable on $[a,b]$ and φ_n is uniformly pointwise bounded, so this provides the bound on $g\varphi_n$ for the Lebesgue Dominated Convergence Theorem).

Theorem 8.5 (continued 3)

Theorem 8.5. Let $1 \le p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof (continued). So for *simple* function f, by $(*)$ we have

$$T(f) = \lim_{n\to\infty} T(\varphi_n) = \lim_{n\to\infty} \left(\int_a^b g\varphi_n \right) = \int_a^b gf.$$

Also, since T is bounded, $\left| \int_a^b gf \right| = |T(f)| \le \|T\|_* \|f\|_p$ for all simple functions f. By Lemma 8.4, $g \in L^q([a,b])$. By Proposition 8.2, the linear functional $L : f \to \int_a^b fg$ is bounded on $L^p([a,b])$.

Theorem 8.5 (continued 3)

Theorem 8.5. Let $1 \leq p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof (continued). So for *simple* function f, by $(*)$ we have

$$T(f) = \lim_{n\to\infty} T(\varphi_n) = \lim_{n\to\infty} \left(\int_a^b g\varphi_n \right) = \int_a^b gf.$$

Also, since T is bounded, $\left| \int_a^b gf \right| = |T(f)| \leq \|T\|_* \|f\|_p$ for all simple functions f. By Lemma 8.4, $g \in L^q([a,b])$. By Proposition 8.2, the linear functional $L : f \to \int_a^b fg$ is bounded on $L^p([a,b])$. Functional L is the same as functional T on all simple functions, simple functions are dense in $L^p([a,b])$ (by Proposition 7.9), and so $L = T$ on all of $L^p([a,b])$ (by Proposition 8.3). That is, $T(f) = \int_a^b gf$ for all $f \in L^p([a,b])$. □

Theorem 8.5 (continued 3)

Theorem 8.5. Let $1 \le p < \infty$. Suppose T is a bounded linear functional on $L^p([a,b])$. Then there is a function $g \in L^q([a,b])$, where q is the conjugate of p, for which $T(f) = \int_{[a,b]} gf$ for all $f \in L^p([a,b])$.

Proof (continued). So for *simple* function f, by $(*)$ we have

$$T(f) = \lim_{n\to\infty} T(\varphi_n) = \lim_{n\to\infty} \left(\int_a^b g\varphi_n \right) = \int_a^b gf.$$

Also, since T is bounded, $\left| \int_a^b gf \right| = |T(f)| \le \|T\|_* \|f\|_p$ for all simple functions f. By Lemma 8.4, $g \in L^q([a,b])$. By Proposition 8.2, the linear functional $L : f \to \int_a^b fg$ is bounded on $L^p([a,b])$. Functional L is the same as functional T on all simple functions, simple functions are dense in $L^p([a,b])$ (by Proposition 7.9), and so $L = T$ on all of $L^p([a,b])$ (by Proposition 8.3). That is, $T(f) = \int_a^b gf$ for all $f \in L^p([a,b])$. □

Riesz Representation Theorem

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_q$.

Proof. By Proposition 8.2, for each $g \in L^q(E)$, $\mathcal{R}_g$ is a bounded linear functional on $L^p(E)$ for which $\|\mathcal{R}_g\|_* = \|g\|_q$. Since integration is linear, for each $g_1, g_2 \in L^q(E)$,

$$\mathcal{R}_{g_1} - \mathcal{R}_{g_2} = \int_E g_1 f - \int_E g_2 f = \int_E (g_1 - g_2) f = \mathcal{R}_{g_1 - g_2}.$$

Riesz Representation Theorem

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_q$.

Proof. By Proposition 8.2, for each $g \in L^q(E)$, $\mathcal{R}_g$ is a bounded linear functional on $L^p(E)$ for which $\|\mathcal{R}_g\|_* = \|g\|_q$. Since integration is linear, for each $g_1, g_2 \in L^q(E)$,

$$\mathcal{R}_{g_1} - \mathcal{R}_{g_2} = \int_E g_1 f - \int_E g_2 f = \int_E (g_1 - g_2) f = \mathcal{R}_{g_1 - g_2}.$$

So if $\mathcal{R}_{g_1} = \mathcal{R}_{g_2}$, then $\mathcal{R}_{g_1-g_2} = 0$ and $\|\mathcal{R}_{g_1-g_2}\|_* = \|g_1 - g_2\|_q = 0$, so $g_1 = g_2$ (a.e.). Therefore, for a bounded linear functional T on $L^p(E)$, there is at most one function $g \in L^q(E)$ for which $\mathcal{R}_g = T$.

Riesz Representation Theorem

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_q$.

Proof. By Proposition 8.2, for each $g \in L^q(E)$, $\mathcal{R}_g$ is a bounded linear functional on $L^p(E)$ for which $\|\mathcal{R}_g\|_* = \|g\|_q$. Since integration is linear, for each $g_1, g_2 \in L^q(E)$,

$$\mathcal{R}_{g_1} - \mathcal{R}_{g_2} = \int_E g_1 f - \int_E g_2 f = \int_E (g_1 - g_2) f = \mathcal{R}_{g_1 - g_2}.$$

So if $\mathcal{R}_{g_1} = \mathcal{R}_{g_2}$, then $\mathcal{R}_{g_1-g_2} = 0$ and $\|\mathcal{R}_{g_1-g_2}\|_* = \|g_1 - g_2\|_q = 0$, so $g_1 = g_2$ (a.e.). Therefore, for a bounded linear functional T on $L^p(E)$, there is at most one function $g \in L^q(E)$ for which $\mathcal{R}_g = T$.

Riesz Representation Theorem (continued 1)

Proof (continued). We now need to show that for each bounded linear functional T on $L^p(E)$, there is a function $g \in L^q(E)$ for which $T = \mathcal{R}_g$. By Theorem 8.5, this holds for $E = [a, b]$. Next, suppose $E = \mathbb{R}$ and let T be a bounded linear functional of $L^p(\mathbb{R})$. For fixed $n \in \mathbb{N}$, define T_n on $L^p([-n, n])$ by $T_n(f) = T(\hat{f})$ for all $f \in L^p([-n, n])$ where $\hat{f}$ is the extension of f to all of $\mathbb{R}$ such that $\hat{f} = 0$ for $x \in \mathbb{R} \setminus [-n, n]$ and $\hat{f}(x) = f(x)$ for $x \in [-n, n]$.

Riesz Representation Theorem (continued 1)

Proof (continued). We now need to show that for each bounded linear functional T on $L^p(E)$, there is a function $g \in L^q(E)$ for which $T = \mathcal{R}_g$. By Theorem 8.5, this holds for $E = [a, b]$. Next, suppose $E = \mathbb{R}$ and let T be a bounded linear functional of $L^p(\mathbb{R})$. For fixed $n \in \mathbb{N}$, define T_n on $L^p([-n, n])$ by $T_n(f) = T(\hat{f})$ for all $f \in L^p([-n, n])$ where $\hat{f}$ is the extension of f to all of $\mathbb{R}$ such that $\hat{f} = 0$ for $x \in \mathbb{R} \setminus [-n, n]$ and $\hat{f}(x) = f(x)$ for $x \in [-n, n]$. Then $\|f\|_p = \|\hat{f}\|_p$, and so $|T_n(f)| = |T(\hat{f})| \le \|T\|_* \|\hat{f}\|_p = \|T\|_* \|f\|_p$ for all $f \in L^p([-n, n])$. So $\|T_n\|_* \le \|T\|_*$. By Theorem 8.5, there is $g_n \in L^q([-n, n])$ for which

$$T_n(f) = \int_{-n}^{n} g_n f \text{ for all } f \in L^p([-n, n]) \text{ and } \|g_n\|_q = \|T_n\|_* \le \|T\|_*. \quad (16)$$

Riesz Representation Theorem (continued 1)

Proof (continued). We now need to show that for each bounded linear functional T on $L^p(E)$, there is a function $g \in L^q(E)$ for which $T = \mathcal{R}_g$. By Theorem 8.5, this holds for $E = [a, b]$. Next, suppose $E = \mathbb{R}$ and let T be a bounded linear functional of $L^p(\mathbb{R})$. For fixed $n \in \mathbb{N}$, define T_n on $L^p([-n, n])$ by $T_n(f) = T(\hat{f})$ for all $f \in L^p([-n, n])$ where $\hat{f}$ is the extension of f to all of $\mathbb{R}$ such that $\hat{f} = 0$ for $x \in \mathbb{R} \setminus [-n, n]$ and $\hat{f}(x) = f(x)$ for $x \in [-n, n]$. Then $\|f\|_p = \|\hat{f}\|_p$, and so $|T_n(f)| = |T(\hat{f})| \leq \|T\|_* \|\hat{f}\|_p = \|T\|_* \|f\|_p$ for all $f \in L^p([-n, n])$. So $\|T_n\|_* \leq \|T\|_*$. By Theorem 8.5, there is $g_n \in L^q([-n, n])$ for which

$$T_n(f) = \int_{-n}^{n} g_n f \text{ for all } f \in L^p([-n, n]) \text{ and } \|g_n\|_q = \|T_n\|_* \leq \|T\|_*. \quad (16)$$

As commented above where we concluded that $g_1 = g_2$ a.e. on E, we now conclude that the restriction of g_{n+1} to $[-n, n]$ equals g_n a.e. on $[-n, n]$. So define g as a measurable function on $\mathbb{R}$ which equals g_n a.e. on $[-n, n]$ for each $n \in \mathbb{N}$.

Riesz Representation Theorem (continued 1)

Proof (continued). We now need to show that for each bounded linear functional T on $L^p(E)$, there is a function $g \in L^q(E)$ for which $T = \mathcal{R}_g$. By Theorem 8.5, this holds for $E = [a, b]$. Next, suppose $E = \mathbb{R}$ and let T be a bounded linear functional of $L^p(\mathbb{R})$. For fixed $n \in \mathbb{N}$, define T_n on $L^p([-n, n])$ by $T_n(f) = T(\hat{f})$ for all $f \in L^p([-n, n])$ where $\hat{f}$ is the extension of f to all of $\mathbb{R}$ such that $\hat{f} = 0$ for $x \in \mathbb{R} \setminus [-n, n]$ and $\hat{f}(x) = f(x)$ for $x \in [-n, n]$. Then $\|f\|_p = \|\hat{f}\|_p$, and so $|T_n(f)| = |T(\hat{f})| \le \|T\|_* \|\hat{f}\|_p = \|T\|_* \|f\|_p$ for all $f \in L^p([-n, n])$. So $\|T_n\|_* \le \|T\|_*$. By Theorem 8.5, there is $g_n \in L^q([-n, n])$ for which

$$T_n(f) = \int_{-n}^{n} g_n f \text{ for all } f \in L^p([-n, n]) \text{ and } \|g_n\|_q = \|T_n\|_* \le \|T\|_*. \quad (16)$$

As commented above where we concluded that $g_1 = g_2$ a.e. on E, we now conclude that the restriction of g_{n+1} to $[-n, n]$ equals g_n a.e. on $[-n, n]$. So define g as a measurable function on $\mathbb{R}$ which equals g_n a.e. on $[-n, n]$ for each $n \in \mathbb{N}$.

Riesz Representation Theorem (continued 2)

Proof (continued). By the definition of T_n and the definition of g_n, along with (16), we have that for all $f \in L^p(\mathbb{R})$ that vanish outside a bounded set,

$$\begin{aligned} T(f) &= \lim_{n\to\infty} T_n(f) \\ &= \lim_{n\to\infty}\left(\int_{-n}^{n} g_n f\right) \text{ by the definition of } g_n \\ &= \int_{\mathbb{R}} gf \text{ by the definition of } g. \end{aligned}$$

So $\lim_{n\to\infty}\left(\int_{-n}^{n}|g|^q\right) = \int_{\mathbb{R}}|g|^q \le (\|T\|_*)^q$ (by (16)) and $g \in L^q(\mathbb{R})$. Since the bounded linear functionals $\mathcal{R}_g$ and T agree on the set of $L^p(\mathbb{R})$ functions which vanish outside a bounded set (which is a set dense in $L^p(\mathbb{R})$), then by Proposition 8.3, $\mathcal{R}_g$ equals T on all of $L^p(\mathbb{R})$.

Riesz Representation Theorem (continued 2)

Proof (continued). By the definition of T_n and the definition of g_n, along with (16), we have that for all $f \in L^p(\mathbb{R})$ that vanish outside a bounded set,

$$\begin{aligned} T(f) &= \lim_{n\to\infty} T_n(f) \\ &= \lim_{n\to\infty} \left(\int_{-n}^{n} g_n f \right) \text{ by the definition of } g_n \\ &= \int_{\mathbb{R}} gf \text{ by the definition of } g. \end{aligned}$$

So $\lim_{n\to\infty} \left(\int_{-n}^{n} |g|^q \right) = \int_{\mathbb{R}} |g|^q \le (\|T\|_*)^q$ (by (16)) and $g \in L^q(\mathbb{R})$. Since the bounded linear functionals $\mathcal{R}_g$ and T agree on the set of $L^p(\mathbb{R})$ functions which vanish outside a bounded set (which is a set dense in $L^p(\mathbb{R})$), then by Proposition 8.3, $\mathcal{R}_g$ equals T on all of $L^p(\mathbb{R})$.

Riesz Representation Theorem (continued 3)

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_p$.

Proof (continued). Finally, consider measurable set E and bounded linear functional T on $L^p(E)$. Define linear functional $\hat{T}$ on $L^p(\mathbb{R})$ as $\hat{T}(f) = T(f|_E)$. Then T is a bounded linear functional on $L^p(\mathbb{R})$.

Riesz Representation Theorem (continued 3)

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_p$.

Proof (continued). Finally, consider measurable set E and bounded linear functional T on $L^p(E)$. Define linear functional $\hat{T}$ on $L^p(\mathbb{R})$ as $\hat{T}(f) = T(f|_E)$. Then T is a bounded linear functional on $L^p(\mathbb{R})$. By above, there is $\hat{g} \in L^q(\mathbb{R})$ for which $\hat{T}$ is represented by integration over $\mathbb{R}$ against $\hat{g}$. Define g to be the restriction of $\hat{g}$ to E. Then $T = \mathcal{R}_g$. □

Riesz Representation Theorem (continued 3)

Riesz Representation Theorem.
Let E be measurable, $1 \leq p < \infty$, and q the conjugate of p. Then for each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by $\mathcal{R}_g(f) = \int_E gf$ for all $f \in L^p(E)$. Then for each bounded linear functional T on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which $\mathcal{R}_g = T$ and $\|T\|_* = \|g\|_p$.

Proof (continued). Finally, consider measurable set E and bounded linear functional T on $L^p(E)$. Define linear functional $\hat{T}$ on $L^p(\mathbb{R})$ as $\hat{T}(f) = T(f|_E)$. Then T is a bounded linear functional on $L^p(\mathbb{R})$. By above, there is $\hat{g} \in L^q(\mathbb{R})$ for which $\hat{T}$ is represented by integration over $\mathbb{R}$ against $\hat{g}$. Define g to be the restriction of $\hat{g}$ to E. Then $T = \mathcal{R}_g$. □