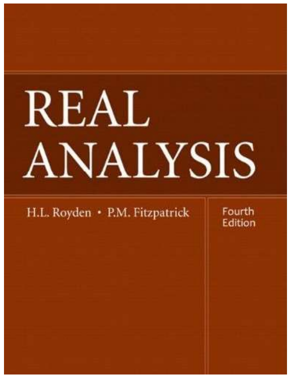
Real Analysis

Chapter 8. The L^p Spaces: Duality and Weak Convergence

8.2. Weak Sequential Convergence in L^p—Proofs of Theorems



Lemma A

Lemma A. The limit of a weakly convergent sequence in $L^p(E)$ is unique, $1 \le p < \infty$.

Proof. Let $\{f_n\} \subset L^p(E)$ and suppose $\{f_n\} \rightharpoonup f$ and $\{f_n\} \rightharpoonup g$. Recall from Hölder's Inequality that the conjugate of $f \in L^p(E)$ is $f^* = \|f\|_p^{1-p}\text{sgn}(f)|f|^{p-1} \in L^q(E)$ and $\int_E ff^* = \|f\|_p$. So $(f-g)^* \in L^q(E)$ and there is $T \in L^p(E)^*$ such that

$$\begin{aligned} T(f) &= \int_E (f-g)f = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} \left(\int_E (f-g) f_n \right) \\ &= T(g) \text{ since } \{f_n\} \rightharpoonup f \text{ and } \{f_n\} \rightharpoonup g \\ &= \int_E (f-g)^* g. \end{aligned}$$

Rearranging, $\int_E (f-g)^* - \int_E (f-g)^* g = 0$, or $\int_E (f-g)^*(f-g) = 0$, or (by the above observation), $\|f-g\|_p = 0$. □

Theorem 8.7

Theorem 8.7. Let f be a measurable set and $1 \le p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\}$ is bounded in $L^p(E)$ and $\|f\|_p \le \liminf \|f_n\|_p$.

Proof. Let q be the conjugate of p and f^* the conjugate function of f as given in Hölder's Inequality. For the claimed inequality, we have

$$\begin{aligned} \int_E f^* f_n &= \int_E |f^* f_n| \text{ since } \text{sgn}(f) = \text{sgn}(f^*) \\ &\le \|f^*\|_q \|f_n\|_p \text{ by Hölder's Inequality} \\ &= \|f_n\|_p \text{ by Hölder's Inequality ("Moreover")} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{f_n\} \rightharpoonup f$ in $L^p(E)$ and $f^* \in L^q(E)$, then by Proposition 8.6 (with $g = f^*$) we have

Theorem 8.7 (continued 1)

Proof (continued).

$$\begin{aligned} \|f\|_p &= \int_E f^* f \text{ by Hölder's Inequality ("Moreover")} \\ &= \lim_{n\to\infty} \int_E f^* f_n \text{ by Proposition 8.6} \\ &\le \liminf \|f_n\|_p \text{ by the inequality established above for all } n \in \mathbb{N}. \end{aligned}$$

Next, we show $\{f_n\}$ is bounded in $L^p(E)$. ASSUME $\{\|f_n\|_p\}$ is unbounded. Then by Problem 8.18 [By possibly taking a subsequence of $\{f_n\}$ and relabeling, we may suppose $\{f_n\| \ge \alpha_n = n3^n$ for all n. By possibly taking a further subsequence and relabeling, we may suppose $\|f_n\|/\alpha_n \to \alpha \in [1, \infty]$. Define $g_n = (\alpha_n/\|f_n\|)f_n$ for each $n \in \mathbb{N}$. Then $\{g_n\}$ converges weakly to αf and $\|g_n\| = n3^n$ for all $n \in \mathbb{N}$.], without loss of generality we suppose

$$\|f_n\|_p = n3^n \text{ for all } n \in \mathbb{N} \qquad (18)$$

(these f_n's are the g_n's of Problem 8.18).

Theorem 8.7 (continued 2)

Proof (continued). We now define a sequence of real numbers $\{\varepsilon_k\}$ inductively, as follows. Define $\varepsilon_1 = 1/3$, and

$$\varepsilon_{n+1} = \begin{cases} 1/3^{n+1} & \text{if } \int_E \left(\sum_{k=1}^n \varepsilon_k (f_k)^*\right) f_{n+1} \geq 0 \\ -1/3^{n+1} & \text{otherwise.} \end{cases}$$

Then

$$\left| \int_E \left(\sum_{k=1}^n \varepsilon_k (f_k)^* \right) f_n \right| = \left| \int_E \left(\sum_{k=1}^{n-1} \varepsilon_k (f_k)^* \right) f_n + \int_E \varepsilon_n (f_n)^* f_n \right|$$

$$= \left| \int_E \left(\sum_{k=1}^{n-1} \varepsilon_k (f_k)^* \right) f_n + \varepsilon_n \|f_n\|_p \right|$$

by Theorem 7.1...

Theorem 8.7 (continued 3)

Proof (continued).

$$\left| \int_E \left(\sum_{k=1}^n \varepsilon_k (f_k)^* \right) f_n \right| = \left| \int_E \left(\sum_{k=1}^{n-1} \varepsilon_k (f_k)^* \right) f_n \right| + |\varepsilon_n| \|f_n\|_p$$

since $\int_E \left(\sum_{k=1}^{n-1} \varepsilon_k (f_k)^* \right) f_n$ and ε_n have the same sign by the definition of ε_n

$$\geq |\varepsilon_n| \|f_n\|_p \text{ dropping the first term}$$

$$= \frac{1}{3^n}(n3^n) = n \text{ by (18).}$$

Also, by the Hölder's Inequality (the "Moreover" part), $\|(f_n)^*\|_q = 1$ and so $\|\varepsilon_n (f_n)^*\|_q = 1/3^n$ for all $n \in \mathbb{N}$. The sequence of partial sums of the series $\sum_{k=1}^\infty \varepsilon_k (f_k)^*$ is a Cauchy sequence in $L^q(E)$. [Let $\varepsilon > 0$. The difference of partial sums is of the form $\sum_{k=m}^{n-1} \varepsilon_k (f_k)^*$ and

Theorem 8.7 (continued 4)

Proof (continued).

$$\left\| \sum_{k=m}^{n-1} \varepsilon_k (f_k)^* \right\|_q \leq \sum_{k=m}^{n-1} \|\varepsilon_k (f_k)^*\|_q = \sum_{k=m}^{n-1} \frac{1}{3^k}$$

$$< \sum_{k=m}^{\infty} \frac{1}{3^k} = \frac{(1/3)^m}{1-(1/3)} = \frac{3}{2}\frac{1}{3^m} = \frac{1}{2 \cdot 3^{m-1}}.$$

For N sufficiently large, with $n > m \geq N$, $\frac{1}{2 \cdot 3^{m-1}}$ can be made less than ε.] Since $L^q(E)$ is complete (by the Riesz-Fischer Theorem), there is $g \in L^q(E)$ with $g = \sum_{k=1}^\infty \varepsilon_k (f_k)^*$. Fix $n \in \mathbb{N}$. then

$$\left| \int_E g f_n \right| = \left| \int_E \left(\sum_{k=1}^\infty \varepsilon_k (f_k)^* \right) f_n \right|$$

Theorem 8.7 (continued 5)

Proof (continued).

$$\left| \int_E g f_n \right| \geq \left| \int_E \left(\sum_{k=1}^n \varepsilon_k (f_k)^* \right) \right| - \left| \int_E \left(\sum_{k=n+1}^\infty \varepsilon_k (f_k)^* \right) f_n \right|$$

by the Triangle Inequality

$$\geq n - \left| \int_E \left(\sum_{k=n+1}^\infty \varepsilon_k (f_k)^* \right) f_n \right| \text{ by the above inequality}$$

$$= n - \left| \sum_{k=n+1}^\infty \int_E \varepsilon_k (f_k)^* f_n \right| \text{ by the Lebesgue Dominated}$$

Convergence Theorem, since $\sum_{k=n+1}^\infty \varepsilon_k (f_k)^* \in L^q(E)$

$$\geq n - \sum_{k=n+1}^\infty |\varepsilon_k| \left| \int_E (f_k)^* f_n \right| \text{ by the Triangle Inequality}$$

Theorem 8.7 (continued 6)

Proof (continued).

$$\begin{aligned}\left|\int_E gf_n\right| &\geq n - \sum_{k=n+1}^{\infty} \frac{1}{3^k}\|(f_k)^*\|_q\|f_n\|_p \text{ by Hölder's Inequality} \\ &= n - \sum_{k=n+1}^{\infty} \frac{1}{3^k} n \text{ since } \|(f_k)^*\|_q = 1 \text{ (by Theorem 7.1} \\ &\quad \text{and since } \|f_n\|_p = n \text{ by above} \\ &= n - \left(\frac{1/3^{n+1}}{1-1/3}\right) n = n - \frac{1}{3^n}\frac{1}{2} n > \frac{n}{2}.\end{aligned}$$

So the sequence of real numbers $\{\int_E gf_n\}$ is not bounded. However, by hypothesis $\{f_n\} \rightharpoonup f$ in $L^p(E)$, so by Proposition 8.6 and Hölder's Inequality,

$$\lim_{n\to\infty} \int_E gf_n = \int_E gf \leq \|g\|_q\|f_n\|_p.$$

Theorem 8.7 (continued 7)

Theorem 8.7. Let f be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\}$ is bounded in $L^p(E)$ and $\|f\|_p \leq \liminf \|f_n\|_p$.

Proof (continued). So $\{\int_E gf_n\}$ converges, and hence is bounded. This is a CONTRADICTION to the assumption that $\{\|f_n\|_p\}$ is not bounded. Hence $\{\|f_n\|_p\}$ is bounded; in other words, $\{f_n\}$ is bounded in $L^p(E)$. □

Corollary 8.8

Corollary 8.8. Let E be a measurable set, $a \leq p < \infty$, and q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ ($\{f_n\} \rightharpoonup f$ in $L^p(E)$) and $\{g_n\}$ converges strongly to g in $L^q(E)$ ($\{g_n\} \to g$ in $L^1(E)$). Then $\lim_{n\to\infty}\left(\int_E g_n f_n\right) = \int_E gf$.

Proof. For each $n \in \mathbb{N}$, by linearity

$$\int_E g_n f_n - \int_E gf = \int_E (g_n - g) f_n + \int_E gf_n - \int_E gf.$$

By Theorem 8.7, $\{f_n\}$ is bounded in $L^p(E)$; that is $\|f_n\|_p \leq C$ for all $n \in \mathbb{N}$, for some given $C \geq 0$. So

$$\left|\int_E g_n f_n - \int_E gf\right| \leq \left|\int_E (g_n - g) f_n\right| + \left|\int_E gf_n - \int_E gf\right| \text{ by the Triangle Inequality}$$

Corollary 8.8 (continued)

Corollary 8.8. Let E be a measurable set, $a \leq p < \infty$, and q the conjugate of p. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$ ($\{f_n\} \rightharpoonup f$ in $L^p(E)$) and $\{g_n\}$ converges strongly to g in $L^q(E)$ ($\{g_n\} \to g$ in $L^1(E)$). Then $\lim_{n\to\infty}\left(\int_E g_n f_n\right) = \int_E gf$.

Proof (continued).

$$\begin{aligned}\left|\int_E g_n f_n - \int_E gf\right| &\leq \|g_n - g\|_q\|f_n\|_p + \left|\int_E gf_n - \int_E gf\right| \\ &\leq C\|g_n - g\|_q + \left|\int_E gf_n - \int_E gf\right|.\end{aligned}$$

Since $\{g_n\} \to g$ in $L^q(E)$, then $\lim_{n\to\infty}\|g_n - g\|_q = 0$, and by Proposition 8.6, $\lim_{n\to\infty}\int_E gf_n - \int_E gf$. The result then follows. □

Proposition 8.9

Proposition 8.9. Let E be a measurable set, $1 \le p < \infty$, and let q be the conjugate of p. Assume $\mathcal{F}$ is a subset of $L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ and let f belong to $L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if $\lim_{n\to\infty}\left(\int_E f_n g\right) = \int_E fg$ for all $g \in \mathcal{F}$.

Proof. If $\{f_n\} \rightharpoonup f$ in $L^p(E)$, then by Proposition 8.6, $\lim_{n\to\infty} \int_E f_n g = \int_E fg$ for all $g \in L^q(E)$, and since $\mathcal{F} \subset L^q(E)$, the result holds.

Next, suppose $\lim_{n\to\infty} \int_E f_n g = \int_E fg$ for all $g \in \mathcal{F}$. Let $g_0 \in L^q(E)$. [We need to show the limit equality holds for g_0.] Let $\varepsilon > 0$. We now find $N \in \mathbb{N}$ such that for all $n \ge N$ we have $\left|\int_E f_n g - \int_E g g_0\right| < \varepsilon$.

Proposition 8.9 (continued 1)

Proof (continued). Notice for any $g \in L^q(E)$ and $n \in \mathbb{N}$ (by Hölder's Inequality):

$$\left|\int_E f_n g_0 - \int_E f g_0\right| = \left|\int_E (f_n - f)(g_0 - g) + \int_E (f_n - f)g\right|$$

$$\le \|f_n - f\|_p \|g - g_0\|_q + \left|\int_E f_n g - \int_E fg\right|.$$

Since $\{f_n\}$ is bounded in $L^p(E)$, then $\|f_n - f\|_p$ is bounded. Since the linear space of $\mathcal{F}$ is dense in $L^q(E)$, there is g in the linear space such that $\|f_n - f\|_p \|g - g_0\|_q < \varepsilon/2$ for all $n \in \mathbb{N}$. Now g is in the linear space of $\mathcal{F}$, and the limit property hold on all of $\mathcal{F}$, so by linearity of integration (and convergence of sequences of real numbers), $\lim_{n\to\infty} \int_E f_n g = \int_E fg$ for g as described.

Proposition 8.9 (continued 2)

Proposition 8.9. Let E be a measurable set, $1 \le p < \infty$, and let q be the conjugate of p. Assume $\mathcal{F}$ is a subset of $L^q(E)$ whose linear span is dense in $L^q(E)$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$ and let f belong to $L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if $\lim_{n\to\infty}\left(\int_E f_n g\right) = \int_E fg$ for all $g \in \mathcal{F}$.

Proof (continued). So there is $N \in \mathbb{N}$ for which $\left|\int_E f_n g - \int_E fg\right| < \varepsilon/2$ for $n \ge N$. Therefore,

$$\left|\int_E f_n g_0 - \int_E g g_0\right| \le \|f_n - f\|_p \|g - g_0\|_q + \left|\int_E f_n g - \int_E fg\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n \ge N.$$

That is, $\lim_{n\to\infty} \int_E f_n g_0 = \int_E f g_0$. □

Theorem 8.10

Theorem 8.10. Let E be a nonmeasurable set and $1 \le p < \infty$, suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ and f belongs to $L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only if for every measurable subset A of E we have $\lim_{n\to\infty} \int_A f_n = \int_A f$. If $p > 1$ (and so $q < \infty$) it is sufficient to consider sets A of finite measure.

Proof. Let

$$\mathcal{F} = \{\chi_A \mid A \text{ is a measurable subset of } E, \chi_A \in L^q(E)\}.$$

The the linear span of $\mathcal{F}$ is the set of all simple functions on E which are in $L^q(E)$. By Theorem 7.9, this span is dense in $L^q(E)$. By Proposition 8.9, $\{f_n\} \rightharpoonup f$ in $L^p(E)$ if and only is $\lim_{n\to} \int_E f_n g = \int_E fg$ for all $g \in \mathcal{F}$; that is, if and only if $\lim_{n\to\infty} \int_E f_n \chi_A = \int_E f \chi_A$ for all measurable $A \subset E$, or if and only if $\int_A f_n = \int_A f$ for all measurable $A \subset E$.

Notice that if $q \ne \infty$ (and $p \ne 1$), then the only characteristic functions in $L^q(E)$ are those of finite support. So if $p = 1$, we need only consider sets A of finite measure. □

Theorem 8.11

Theorem 8.11. Let $[a, b]$ be a closed, bounded interval and $1 < p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p[a, b]$ and $f \in L^p[a, b]$. Then $\{f_n\| \rightharpoonup f$ in $L^p[a, b]$ if and only if $\lim_{n\to\infty} \int_a^x f_n = \int_a^x f$ for all $x \in [a, b]$.

Proof. Let $\mathcal{F} = \{\chi_{[a,x]} \mid x \in [a, b] \text{ and } \chi_{[a,x]} \in L^q([a, b])\}$. Then the linear span of $\mathcal{F}$ is the set of all step functions on E which are in $L^q([a, b])$. By Theorem 7.10, this span is dense in $L^q([a, b])$. By Proposition 8.9, $\{f_n\} \rightharpoonup f$ in $L^p([a, b])$ if and only if $\lim_{n\to\infty} \int_E f_n g - \int_E fg$ for all $g \in \mathcal{F}$; that is, if and only if

$$\lim_{n\to\infty} \int_{[a,b]} f_n \chi_{[a,x]} = \int_{[a,b]} f \chi_{[a,x]} \text{ for all } x \in [a, b],$$

or if and only if

$$\lim_{n\to\infty} \int_{[a,x]} f_n = \int_{[a,x]} f \text{ for all } x \in [a, b].$$

□

Theorem 8.12

Theorem 8.12. Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\}$ is a bounded sequence in $L^p(E)$ that converges pointwise a.e. on E to f. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$.

Proof. By Fatou's Lemma, since $\{f_n\} \to f$ pointwise, $\int_E |f|^p \le \liminf \int_E |f|^p < \infty$ since $\{f_n\}$ is bounded in $L^p(E)$. So $f \in L^p(E)$. Let $A \subset E$ be measurable with $m(A) < \infty$. By Corollary 7.2, the sequence $\{f_n\}$ is uniformly integrable over E. By the Vitali Convergence Theorem, $\lim_{n\to\infty} \int_A f_n = \int_A f$ (since $m(A) < \infty$). So by Theorem 8.11, $\{f_n\} \rightharpoonup f$ weakly in $L^p(E)$. □

Radon-Riesz Theorem

The Radon-Riesz Theorem.
Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Proof for $n = 2$. Let $n = 2$ (the general proof is given in the supplement to the notes for Section 8.2). Let $\{f_n\}$ be a sequence in $L^2(E)$ such that $\{f_n\} \rightharpoonup f$. For each $n \in \mathbb{N}$, by linearity of integration,

$$\|f_n - f\|_2^2 = \int_E |f_n - f|^2 = \int_E (f_n - f)^2 = \int_E (f_n^2 - 2f_n f + f^2)$$

$$= \int_E |f_n|^2 - 2\int_E f_n f + \int_E |f|^2.$$

We hypothesize that $\{f_n\} \rightharpoonup f$ in $L^2(E)$ and $f \in L^2(E)$, so by the Riesz Representation Theorem for T_f and the weak convergence hypothesis, we have $\lim \int_E f_n f = \int_E f^2$. So

$$\lim_{n\to\infty} \|f_n - f\|_2^2 = \lim_{n\to\infty} \left(\int_E |f_n|^2 - 2\int_E f_n f + \int_E |f|^2 \right)$$

Radon-Riesz Theorem (continued)

The Radon-Riesz Theorem.
Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Proof for $n = 2$ (continued).

$$\lim_{n\to\infty} \|f_n - f\|_2^2 = \lim_{n\to\infty} \int_E |f_n|^2 - 2\lim_{n\to\infty} \int_E f_n f + \lim_{n\to\infty} \int_E |f|^2$$

and hence

$$\lim_{n\to\infty} \|f_n - f\|_2^2 = \lim_{n\to\infty} \int_E |f_n|^2 - 2\int_E |f|^2 + \int_E |f|^2$$

$$= \lim_{n\to\infty} \int_E |f_n|^2 - \int_E |f|^2 = \lim_{n\to\infty} \|f_n\|_2^2 - \|f\|^2.$$

So $\{f_n\} \to f$ with respect to the $L^2(E)$ norm if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. □

Corollary 8.13

Corollary 8.13. Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then a subsequence of $\{f_n\}$ converges strongly in $L^p(E)$ to f if and only if $\|f\|_p = \liminf \|f_n\|_p$.

Proof. If $\|f\|_p = \liminf \|f_n\|_p$, then there is a subsequence $\{f_{n_k}\}$ for which $\lim_{k\to\infty} \|f_{n_k}\|_p = \|f\|_p$. Since a subsequence of a weakly convergent sequence if weakly convergent, we can apply the Radon Riesz Theorem to $\{f_{n_k}\}$ and conclude that $\{f_{n_k}\} \to f$ in $L^p(E)$ (i.e., "strongly").

Conversely, if there is a subsequence $\{f_{n_k}\}$ that converges (strongly) to f in $L^p(E)$, then $\lim_{k\to\infty} \|f_{n_k}\|_p = \|f\|_p$. So $\liminf \|f_n\|_p \le \lim \|f_{n_k}\|_p = \|f\|_p$. By Theorem 8.7, since $\{f_n\} \rightharpoonup f$ in $L^p(E)$ then $\|f\|_p \le \liminf \|f_n\|_p$. The result now follows. □