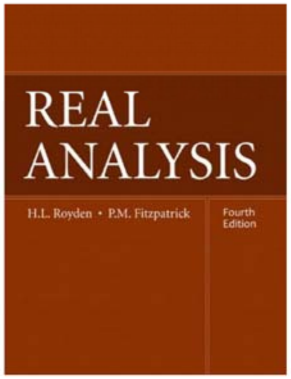
Real Analysis

Chapter 8. The L^p Spaces: Duality and Weak Convergence

8.3. Weak Sequential Compactness—Proofs of Theorems



Helly's Theorem

Helly's Theorem. Let X be a separable normed linear space and $\{T_n\}$ a sequence in its dual space X^* that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k\to\infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

Proof. Let $\{f_j\}_{j=1}^\infty$ be a countable dense subset of X that is dense in X. We now apply the sequence $\{T_n\}$ to the sequence $\{f_j\}$. The sequence of real numbers $\{T_n(f_1)\}$ is bounded by hypothesis. So, by the Bolzano-Weierstrass Theorem, there is a strictly increasing sequence of natural numbers (indices) $\{s(1,n)\}_{n=1}^\infty$ and a number $a_1 \in \mathbb{R}$ for which $\lim_{n\to\infty} T_{s(1,n)}(f_1) = a_1$. By hypothesis, $\{T_{s(1,n)}(f_2)\}_{n=1}^\infty$ is bounded and so again by Bolzano-Weierstrass there is a subsequence $\{s(2,n)\}_{n=1}^\infty$ of $\{s(1,n)\}_{n=1}^\infty$ and $a_2 \in \mathbb{R}$ for which $\lim_{n\to\infty} T_{s(2,n)}(f_2) = a_2$.

Helly's Theorem (continued 1)

Proof (continued). We inductively continue this process to produce a countable collection of strictly increasing sequences of natural numbers $\{\{s(j,n)\}_{n=1}^\infty\}_{j=1}^\infty$ and a sequence $\{a_j\}_{j=1}^\infty$ of real numbers such that $\{s(j+1,n)\}_{n=1}^\infty$ is a subsequence of $\{s(j,n)\}_{n=1}^\infty$ for all $j \in \mathbb{N}$, and $\lim_{n\to\infty} T_{s(j,n)}(f_j) = a_j$ for all $j \in \mathbb{N}$.

For each $k \in \mathbb{N}$, define $n_k = s(k,k)$. Then for each $j \in \mathbb{N}$, $\{n_k\}_{k=j}^\infty$ is a subsequence of $\{s(j,k)\}_{k=1}^\infty$ (because of the nestedness of the sequences in terms of the j parameter). So $\lim_{k\to\infty} T_{n_k}(f_j) = a_j$ for all $j \in \mathbb{N}$.

Since$\{T_{n_k}\}$ is bounded in X^* (that is, $\|T_{n_k}\|_* \leq M$ for all T_{n_k}), $\{T_{n_k}(f)\}$ is convergent (and therefore Cauchy) for each $f \in \{f_j\}$, and since $\{f_j\}$ is dense in X, then $\{T_{n_k}(f)\}$ is Cauchy for all $f \in X$ (by Problem 8.35). So, for all $f \in X$, $\{T_{n_k}(f)\}_{k=1}^\infty$ converges to some real number, say $\lim_{k\to\infty} T_{n_k}(f) = T(f)$. We now show that T is a bounded linear functional.

Helly's Theorem (continued 2)

Helly's Theorem. Let X be a separable normed linear space and $\{T_n\}$ a sequence in its dual space X^* that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k\to\infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

Proof (continued). First,

$$T(\alpha f + \beta g) = \lim_{k\to\infty} T_{n_k}(\alpha f + \beta g) = \lim_{k\to\infty} (\alpha T_{n_k}(f) + \beta T_{n_k}(g))$$

$$= \alpha \lim_{k\to\infty} T_{n_k}(f) + \beta \lim_{k\to\infty} T_{n_k}(g) = \alpha T(f) + \beta T(g).$$

Next, $|T_{n_k}(f)| \leq M\|f\|$ for all $k \in \mathbb{N}$ and for all $f \in X$, so $|T(f)| = \lim_{k\to\infty} T_{n_k}(f)| \leq M\|f\|$ for all $f \in X$, and so $\|T\|_* \leq M$ and T is unbounded. □

Theorem 8.14

Theorem 8.14. Let E be a measurable set and $1 < p < \infty$. Then every founded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Proof. Let q be the conjugate of p. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$. Define $X = L^q(E)$. For $n \in \mathbb{N}$, define T_n on X as $T_n(g) = \int_E f_n g$ for $g \in X = L^q(E)$. By Proposition 8.2 (with p and q interchanged) T_n is a bounded linear functional on X and $\|T_n\|_* = \|f_n\|_p$. Since $\{f_n\}$ is bounded in $L^p(E)$, then $\{T_n\}$ is bounded in X^*.

Moreover, by Theorem 7.11, since $1 < q < \infty$, $X = L^q(E)$ is separable. So, by Helly's Theorem, there is a subsequence $\{T_{n_k}\}$ and $T \in X^*$ such that $\lim_{k\to\infty} T_{n_k}(g) = T(g)$ for all $g \in X = L^q(E)$. By the Riesz Representation Theorem (with p and q interchanged), there is $f \in L^p(E)$ for which $T(g) = \int_E fg$ for all $g \in X = L^q(E)$.

Theorem 8.14 (continued)

Theorem 8.14. Let E be a measurable set and $1 < p < \infty$. Then every founded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Proof (continued). But then we have that

$$\lim_{k\to\infty} T_{n_k}(g) = \lim_{k\to\infty} \int_E f_{n_k} g = T(g) = \int_E fg \text{ for all } g \in X = L^q(E).$$

By Proposition 8.6, $\{f_{n_k}\}$ converges weakly to f in $L^p(E)$. □

Theorem 8.15

Theorem 8.15. Let E be a measurable set and $1 < p < \infty$. Then $\{f \in L^p(E) \mid \|f\|_p \le 1\}$ is weakly sequentially compact in $L^p(E)$.

Proof. Let $\{f_n\}$ be a sequence in $L^p(E)$ for which $\|f\|_p \le 1$ for all $n \in \mathbb{N}$. By Theorem 8.14, there is a subsequence $\{f_{n_k}\}$ which converges weakly to $f \in L^p(E)$. Moreover, $\|f\|_p \le 1$, since by Theorem 8.7, $\|f\|_p \le \liminf \|f_n\|_p \le 1$. Therefore $f \in \{f \in L^p(E) \mid \|f\|_p \le 1\}$, and the set is sequentially compact. □