Chapter 8. The $L^p$ Spaces: Duality and Weak Convergence
8.3. Weak Sequential Compactness—Proofs of Theorems
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Helly’s Theorem

Helly’s Theorem. Let $X$ be a separable normed linear space and $\{T_n\}$ a sequence in its dual space $X^*$ that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k \to \infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

Proof. Let $\{f_j\}_{j=1}^{\infty}$ be a countable dense subset of $X$ that is dense in $X$. We now apply the sequence $\{T_n\}$ to the sequence $\{f_j\}$. 
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Proof. Let $\{f_j\}_{j=1}^{\infty}$ be a countable dense subset of $X$ that is dense in $X$. We now apply the sequence $\{T_n\}$ to the sequence $\{f_j\}$. The sequence of real numbers $\{T_n(f_1)\}$ is bounded by hypothesis. So, by the Bolzano-Weierstrass Theorem, there is a strictly increasing sequence of natural numbers (indices) $\{s(1, n)\}_{n=1}^{\infty}$ and a number $a_1 \in \mathbb{R}$ for which $\lim_{n \to \infty} T_{s(1, n)}(f_1) = a_1$. 
Helly’s Theorem. Let $X$ be a separable normed linear space and $\{T_n\}$ a sequence in its dual space $X^*$ that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k \to \infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

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Helly’s Theorem. Let $X$ be a separable normed linear space and $\{T_n\}$ a sequence in its dual space $X^*$ that is bounded. That is, there exists $M \geq 0$ for which $|T_n(f)| \leq M\|f\|$ for all $f \in X$, for all $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which $\lim_{k \to \infty} T_{n_k}(f) = T(f)$ for all $f \in X$.

Proof. Let $\{f_j\}_{j=1}^\infty$ be a countable dense subset of $X$ that is dense in $X$. We now apply the sequence $\{T_n\}$ to the sequence $\{f_j\}$. The sequence of real numbers $\{T_n(f_1)\}$ is bounded by hypothesis. So, by the Bolzano-Weierstrass Theorem, there is a strictly increasing sequence of natural numbers (indices) $\{s(1, n)\}_{n=1}^\infty$ and a number $a_1 \in \mathbb{R}$ for which $\lim_{n \to \infty} T_{s(1,n)}(f_1) = a_1$. By hypothesis, $\{T_{s(1,n)}(f_2)\}_{n=1}^\infty$ is bounded and so again by Bolzano-Weierstrass there is a subsequence $\{s(2, n)\}_{n=1}^\infty$ of $\{s(1, n)\}_{n=1}^\infty$ and $a_2 \in \mathbb{R}$ for which $\lim_{n \to \infty} T_{s(2,n)}(f_2) = a_2$. 
Helly’s Theorem (continued 1)

**Proof (continued).** We inductively continue this process to produce a countable collection of strictly increasing sequences of natural numbers \(\{\{s(j, n)\}_{n=1}^{\infty}\}_{j=1}^{\infty}\) and a sequence \(\{a_j\}_{j=1}^{\infty}\) of real numbers such that \(\{s(j + 1, n)\}_{n=1}^{\infty}\) is a subsequence of \(\{s(j, n)\}_{n=1}^{\infty}\) for all \(j \in \mathbb{N}\), and \(\lim_{n \to \infty} T_{s(j,n)}(f_j) = a_j\) for all \(j \in \mathbb{N}\).

For each \(k \in \mathbb{N}\), define \(n_k = s(k, k)\). Then for each \(j \in \mathbb{N}\), \(\{n_k\}_{k=j}^{\infty}\) is a subsequence of \(\{s(j, k)\}_{k=1}^{\infty}\) (because of the nestedness of the sequences in terms of the \(j\) parameter). So \(\lim_{k \to \infty} T_{n_k}(f_j) = a_j\) for all \(j \in \mathbb{N}\).
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\[
\{\{s(j, n)\}_{n=1}^{\infty}\}_{j=1}^{\infty} \quad \text{and a sequence } \{a_j\}_{j=1}^{\infty} \text{ of real numbers such that }
\{s(j + 1, n)\}_{n=1}^{\infty} \text{ is a subsequence of } \{s(j, n)\}_{n=1}^{\infty} \text{ for all } j \in \mathbb{N}, \text{ and }
\lim_{n \to \infty} T_{s(j, n)}(f_j) = a_j \text{ for all } j \in \mathbb{N}.
\]

For each \( k \in \mathbb{N} \), define \( n_k = s(k, k) \). Then for each \( j \in \mathbb{N} \), \( \{n_k\}_{k=j}^{\infty} \) is a subsequence of \( \{s(j, k)\}_{k=1}^{\infty} \) (because of the nestedness of the sequences in terms of the \( j \) parameter). So \( \lim_{k \to \infty} T_{n_k}(f_j) = a_j \) for all \( j \in \mathbb{N} \).

Since \( \{T_{n_k}\} \) is bounded in \( X^* \) (that is, \( \|T_{n_k}\|_* \leq M \) for all \( T_{n_k} \)), \( \{T_{n_k}(f)\} \) is convergent (and therefore Cauchy) for each \( f \in \{f_j\} \), and since \( \{f_j\} \) is dense in \( X \), then \( \{T_{n_k}(f)\} \) is Cauchy for all \( f \in X \) (by Problem 8.35). So, for all \( f \in X \), \( \{T_{n_k}(f)\}_{k=1}^{\infty} \) converges to some real number, say \( \lim_{k \to \infty} T_{n_k}(f) = T(f) \). We now show that \( T \) is a bounded linear functional.
Proof (continued). We inductively continue this process to produce a countable collection of strictly increasing sequences of natural numbers \( \{s(j, n)\}_{n=1}^{\infty} \) and a sequence \( \{a_j\}_{j=1}^{\infty} \) of real numbers such that \( \{s(j + 1, n)\}_{n=1}^{\infty} \) is a subsequence of \( \{s(j, n)\}_{n=1}^{\infty} \) for all \( j \in \mathbb{N} \), and \( \lim_{n \to \infty} T_{s(j, n)}(f_j) = a_j \) for all \( j \in \mathbb{N} \).

For each \( k \in \mathbb{N} \), define \( n_k = s(k, k) \). Then for each \( j \in \mathbb{N} \), \( \{n_k\}_{k=j}^{\infty} \) is a subsequence of \( \{s(j, k)\}_{k=1}^{\infty} \) (because of the nestedness of the sequences in terms of the \( j \) parameter). So \( \lim_{k \to \infty} T_{n_k}(f_j) = a_j \) for all \( j \in \mathbb{N} \).

Since \( \{T_{n_k}\} \) is bounded in \( X^* \) (that is, \( \|T_{n_k}\|_* \leq M \) for all \( T_{n_k} \) ), \( \{T_{n_k}(f)\} \) is convergent (and therefore Cauchy) for each \( f \in \{f_j\} \), and since \( \{f_j\} \) is dense in \( X \), then \( \{T_{n_k}(f)\} \) is Cauchy for all \( f \in X \) (by Problem 8.35). So, for all \( f \in X \), \( \{T_{n_k}(f)\}_{k=1}^{\infty} \) converges to some real number, say \( \lim_{k \to \infty} T_{n_k}(f) = T(f) \). We now show that \( T \) is a bounded linear functional.
Helly’s Theorem. Let \( X \) be a separable normed linear space and \( \{ T_n \} \) a sequence in its dual space \( X^* \) that is bounded. That is, there exists \( M \geq 0 \) for which \( |T_n(f)| \leq M\|f\| \) for all \( f \in X \), for all \( n \in \mathbb{N} \). Then there is a subsequence \( \{ T_{n_k} \} \) of \( \{ T_n \} \) and \( T \in X^* \) for which \( \lim_{k \to \infty} T_{n_k}(f) = T(f) \) for all \( f \in X \).

Proof (continued). First,

\[
T(\alpha f + \beta g) = \lim_{k \to \infty} T_{n_k}(\alpha f + \beta g) = \lim_{k \to \infty} (\alpha T_{n_k}(f) + \beta T_{n_k}(g))
\]

\[
= \alpha \lim_{k \to \infty} T_{n_k}(f) + \beta \lim_{k \to \infty} T_{n_k}(g) = \alpha T(f) + \beta T(g).
\]

Next, \( |T_{n_k}(f)| \leq M\|f\| \) for all \( k \in \mathbb{N} \) and for all \( f \in X \), so \( |T(f)| = \lim_{k \to \infty} T_{n_k}(f)| \leq M\|f\| \) for all \( f \in X \), and so \( \|T\|_* \leq M \) and \( T \) is unbounded.
Theorem 8.14. Let $E$ be a measurable set and $1 < p < \infty$. Then every founded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

Proof. Let $q$ be the conjugate of $p$. Let $\{f_n\}$ be a bounded sequence in $L^p(E)$. Define $X = L^q(E)$. 

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Proof. Let $q$ be the conjugate of $p$. Let $\{f_n\}$ be a bounded sequence in 
$L^p(E)$. Define $X = L^q(E)$. For $n \in \mathbb{N}$, define $T_n$ on $X$ as 
$T_n(g) = \int_E f_n g$ 
for $g \in X = L^q(E)$. By Proposition 8.2 (with $p$ and $q$ interchanged) $T_n$ is 
a bounded linear functional on $X$ and $\|T_n\|_* = \|f_n\|_p$. Since $\{f_n\}$ is 
bounded in $L^p(E)$, then $\{T_n\}$ is bounded in $X^*$. 

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Moreover, by Theorem 7.11, since $1 < q < \infty$, $X = L^q(E)$ is separable. So, by Helly’s Theorem, there is a subsequence $\{T_{n_k}\}$ and $T \in X^*$ such that $\lim_{k \to \infty} T_{n_k}(g) = T(g)$ for all $g \in X = L^q(E)$. By the Riesz Representation Theorem (with $p$ and $q$ interchanged), there is $f \in L^p(E)$ for which $T(g) = \int_E fg$ for all $g \in X = L^q(E)$.
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Moreover, by Theorem 7.11, since $1 < q < \infty$, $X = L^q(E)$ is separable. So, by Helly's Theorem, there is a subsequence $\{T_{n_k}\}$ and $T \in X^*$ such that $\lim_{k \to \infty} T_{n_k}(g) = T(g)$ for all $g \in X = L^q(E)$. By the Riesz Representation Theorem (with $p$ and $q$ interchanged), there is $f \in L^p(E)$ for which $T(g) = \int_E fg$ for all $g \in X = L^q(E)$. 

Theorem 8.14 (continued)

**Theorem 8.14.** Let $E$ be a measurable set and $1 < p < \infty$. Then every founded sequence in $L^p(E)$ has a subsequence that converges weakly in $L^p(E)$ to a function in $L^p(E)$.

**Proof (continued).** But then we have that

$$\lim_{k \to \infty} T_{n_k}(g) = \lim_{k \to \infty} \int_E f_{n_k}g = T(g) = \int_E fg \text{ for all } g \in X = L^q(E).$$

By Proposition 8.6, $\{f_{n_k}\}$ converges weakly to $f$ in $L^p(E)$.
Theorem 8.15. Let $E$ be a measurable set and $1 < p < \infty$. Then 
\[ \{ f \in L^p(E) \mid \|f\|_p \leq 1 \} \] 
is weakly sequentially compact in $L^p(E)$.

Proof. Let \( \{ f_n \} \) be a sequence in $L^p(E)$ for which \( \|f\|_p \leq 1 \) for all \( n \in \mathbb{N} \). By Theorem 8.14, there is a subsequence \( \{ f_{n_k} \} \) which converges weakly to $f \in L^p(E)$. 
Theorem 8.15. Let $E$ be a measurable set and $1 < p < \infty$. Then \( \{ f \in L^p(E) \mid \|f\|_p \leq 1 \} \) is weakly sequentially compact in $L^p(E)$.

Proof. Let \( \{ f_n \} \) be a sequence in $L^p(E)$ for which $\|f\|_p \leq 1$ for all $n \in \mathbb{N}$. By Theorem 8.14, there is a subsequence \( \{ f_{n_k} \} \) which converges weakly to $f \in L^p(E)$. Moreover, $\|f\|_p \leq 1$, since by Theorem 8.7, $\|f\|_p \leq \lim \inf \|f_n\|_p \leq 1$. Therefore $f \in \{ f \in L^p(E) \mid \|f\|_p \leq 1 \}$, and the set is sequentially compact.
**Theorem 8.15.** Let $E$ be a measurable set and $1 < p < \infty$. Then \( \{ f \in L^p(E) \mid \| f \|_p \leq 1 \} \) is weakly sequentially compact in $L^p(E)$.

**Proof.** Let \( \{ f_n \} \) be a sequence in $L^p(E)$ for which $\| f \|_p \leq 1$ for all $n \in \mathbb{N}$. By Theorem 8.14, there is a subsequence \( \{ f_{n_k} \} \) which converges weakly to $f \in L^p(E)$. Moreover, $\| f \|_p \leq 1$, since by Theorem 8.7, $\| f \|_p \leq \lim \inf \| f_n \|_p \leq 1$. Therefore $f \in \{ f \in L^p(E) \mid \| f \|_p \leq 1 \}$, and the set is sequentially compact. \qed