Real Analysis 1, MATH 5210

Homework 9, Section 4.3, Solutions

Due Monday November 17, 2014 at 1:30

Prove each of the following.

Problem 4.17. Let E be a set of measure zero and define $f = \infty$ on E. Prove that $\int_E f = 0$.

Problem 4.26. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Problem 4.27. Prove the Generalized Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions in E, then

$$\int_E (\liminf f_n) \leq \liminf \left(\int_E f_n \right).$$

Problem 4.24. Bonus.

Let f be a nonnegative measurable function on E.

(i) Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on E, each of finite support, which converges pointwise on E to f.

(ii) Show that $\int_E f = \sup\{\int_E \varphi \mid \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f \text{ on } E\}$.

HINT for (i). Partition the interval $[0, n]$ of y-values into $n2^n$ pieces, each of length 2^{-n}. Do this in such a way that an increase in n produces a refinement of the partition (this yields an increasing sequence). Define φ_n as given in the following pictures. Notice this means that φ_n is nonnegative, bounded above by n and is nonzero on a set of finite measure (of measure less than or equal to $2n$). **HINT for (ii).** Apply the Monotone Convergence Theorem to the sequence φ_n of part (i) and use Monotonicity of Integration (Theorem 4.10).