

# Real Analysis 1, MATH 5210, Fall 2016

## Homework 2, Riemann Integrals

Due Friday, September 2, at 1:30

**Write in complete sentences!!!** *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!** Assume throughout these exercises that function  $f$  is a bounded defined on  $[a, b]$

**RI.A. (a)** Let  $X$  and  $Y$  be nonempty sets of real numbers such that for any  $x \in X$  and  $y \in Y$  we have  $x \leq y$ . Prove that  $\sup(X)$  and  $\inf(Y)$  are finite and that  $\sup(X) \leq \inf(Y)$ . HINT: Use the definition of  $\sup(X) = \text{lub}(X)$  and  $\inf(Y) = \text{glb}(Y)$ .

**(b)** For  $X$  and  $Y$  as above, prove that  $\sup(X) = \inf(Y)$  if and only if for all  $\varepsilon > 0$  there exist  $x(\varepsilon) \in X$  and  $y(\varepsilon) \in Y$  such that  $y(\varepsilon) - x(\varepsilon) < \varepsilon$ . HINT: To show the existence of  $x(\varepsilon)$  and  $y(\varepsilon)$ , use Exercise 0.3 (and the obvious related result for  $\text{glb}$ ).

**RI.B. (a)** Let  $P = \{x_0, x_1, \dots, x_n\}$  be a partition of  $[a, b]$  and let

$$Q = P \cup \{x'\} = \{x_0, x_1, \dots, x_k, x', x_{k+1}, \dots, x_n\}$$

( $Q$  is a *one point refinement* of  $P$ ). Prove that  $\underline{S}(f; P) \leq \underline{S}(f; Q)$ . You may assume that  $A \subset B$  implies  $\inf(B) \leq \inf(A)$ .

**(b)** Let  $P = \{x_0, x_1, \dots, x_n\}$  be a partition of  $[a, b]$  and let  $Q$  be a partition of  $[a, b]$  such that  $P \subset Q$  ( $Q$  is called a *refinement* of  $P$ ). Prove that  $\underline{S}(f; P) \leq \underline{S}(f; Q)$ . NOTE: A similar argument shows that  $\overline{S}(f; P) \geq \overline{S}(f; Q)$ .

**RI.C. (a)** If  $P$  and  $Q$  are partitions of  $[a, b]$ , prove that  $\underline{S}(f; P) \leq \overline{S}(f; Q)$ . HINT: Consider the partition  $P \cup Q$ .

**(b)** Let

$$\underline{S}(f) = \sup\{\underline{S}(f; P) \mid P \text{ is a partition of } [a, b]\},$$

$$\overline{S}(f) = \inf\{\overline{S}(f; P) \mid P \text{ is a partition of } [a, b]\}.$$

Then  $\underline{S}(f)$  and  $\overline{S}(f)$  are finite, and  $\underline{S}(f) \leq \overline{S}(f)$ . HINT: Use Exercise RI.A(a) and part (a) of this exercise.