Real Analysis 1, MATH 5210, Fall 2016

Homework 5, Lebesgue Outer Measure, DRAFT

Due Friday, September 23, at 1:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

2.2.A. Prove that if G is a bounded open set, then $m^*(G)$ equals the sum of the lengths of its constituent countable disjoint open intervals. HINT: Use Theorem 0.7 to write $G = \cup_{n=1}^{\infty} I_n$ where each I_n is a bounded open interval and prove $m^*(G) = \sum_{n=1}^{\infty} \ell(I_n)$.

2.2.B. Prove that if we define the outer measure of a set $E \subset \mathbb{R}$ as

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} m^*(G_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} G_n \text{ and each } G_n \text{ is a bounded open set} \right\},$$

then for all $E \subset \mathbb{R}$ we have $\lambda^*(E) = m^*(E)$. NOTE: This shows that we could use bounded open *sets* to define outer measure instead of bounded open *intervals* (some texts use this approach to outer measure; for example, A.M. Bruckner, J.B. Bruckner, and B.S. Thomson's *Real Analysis*, Prentice Hall (1997)). HINT: Show

$$\left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is a bounded open interval} \right\}$$
$$= \left\{ \sum_{n=1}^{\infty} m^*(G_n) \,\middle|\, E \subset \cup_{n=1}^{\infty} G_n \text{ and each } G_n \text{ is a bounded open set} \right\}.$$

2.10 Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$ and $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.2.C. BONUS. Let G be an open subset of $[a, b]$. Prove that $m^*([a, b] \setminus G) = b - a - m^*(G)$. NOTE: This is somewhat related to the development of "inner measure."