Real Analysis 1, MATH 5210, Fall 2016

Homework 6, The σ-Algebra of Lebesgue Measurable Sets

Due Friday, September 30, at 1:30

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!** You may assume countable additivity for measurable sets in these exercises.

2.3.A. (a) Prove that if G is a bounded open set, then $m^*(G)$ equals the sum of the lengths of its "constituent" countable disjoint open intervals (these intervals are the *connected components* of G). HINT: Use Theorem 0.7 to write $G = \cup_{n=1}^{\infty} I_n$ where each I_n is a bounded open interval and prove $m^*(G) = \sum_{n=1}^{\infty} \ell(I_n)$.

(b) Prove that if G is an open set, then $m^*(G)$ equals the sum of the lengths of its constituent countable disjoint open intervals. HINT: Consider two cases. First consider the case where each constituent open interval of G is bounded and second consider the case where G has an unbounded constituent open interval.

2.14. Prove that if a set E has finite positive outer measure, then there is a bounded subset of E that also has positive outer measure. HINT: Consider the contrapositive.

2.15. Prove that if E is a measurable set with finite outer measure and $\varepsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has outer measure at most ε. HINT: Cut set E into bounded pieces and a single unbounded piece which is small in measure.

2.3.B. Bonus Let E be a measurable set with finite positive outer measure. Prove for any ε with $0 < \varepsilon < 1$ that there is a bounded interval I such that $\varepsilon m^*(I) \leq m^*(E \cap I) \leq m^*(I)$. HINT: Use the definition of outer measure and Theorem 0.3 with $\varepsilon > 0$ of the theorem replaced with $(1/\varepsilon - 1)m^*(E) > 0$. You will need Theorem 0.3 and countable additivity. Beware when you use bounded open intervals and disjoint open intervals.