Real Analysis 1, MATH 5210, Fall 2018

Homework 8, Sums, Products, and Compositions, Solutions

3.3. Suppose a function f has a measurable domain E and is continuous except at a finite number of points. Is f necessarily measurable? If so, then prove it. If not, then give an example.

Proof. YES! Let the points of discontinuity be (in order) $\{x_1, x_2, \ldots, x_n\}$. Then f is continuous on $X_k = E \cap (x_k, x_{k+1})$ for $k = 1, 2, \ldots, n-1$, continuous on $X_0 = E \cap (-\infty, x_1)$, and continuous on $X_n = E \cap (x_n, \infty)$. Let $c \in \mathbb{R}$. then

$$X_k' = \{x \in X_k \mid f(x) > c\} \in \mathcal{M} \text{ for } k = 0, 1, 2, \ldots, n$$

by Proposition 3.3. So

$$\{x \in E \mid f(x) > c\} = (\cup_{k=0}^{n} X_k') \cup A$$

where A is some subset of $\{x_1, x_2, \ldots, x_n\}$. Since each $X_k' \in \mathcal{M}$ and $A \in \mathcal{M}$, then $\{x \in E \mid f(x) > c\} \in \mathcal{M}$ and f is measurable by Proposition 3.1(i). ∎

3.4. Suppose f is a real-valued function defined on $\mathbb{R}$ such that $f^{-1}(\{c\}) \in \mathcal{M}$ for each number c. Is f necessarily measurable? If so, then prove it. If not, then give an example.

Solution. NO! Consider

$$f(x) = \begin{cases} e^x & \text{for } x \notin P \\ -e^x & \text{for } x \in P \end{cases}$$

where P is the nonmeasurable Vitali set. Then for each $c \in \mathbb{R}$, $f^{-1}(\{c\})$ is either empty or a singleton and so $f^{-1}(\{c\}) \in \mathcal{M}$. However, $\{x \in \mathbb{R} \mid f(x) < 0\} = P \notin \mathcal{M}$. So f is not measurable by Proposition 3.1(iii). □

3.6. Let f be a function with measurable domain D. Prove that f is measurable if and only if the function g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Proof. Suppose f is measurable with domain D. Then for any $c > 0$, $\{x \in D \mid f(x) \geq c\} = \{x \in \mathbb{R} \mid g(x) \geq c\}$ is measurable by the definition of "f is measurable" (Proposition 3.1(b)). Similarly, for any $c \leq 0$, $\{x \in \mathbb{R} \mid g(x) \geq c\} = \{x \in D \mid f(x) \geq c\} \cup (\mathbb{R} \setminus D\}$ is measurable since f is measurable (Proposition 3.1(b)) and $\mathbb{R} \setminus D$ is measurable since D is measurable and $\mathcal{M}$ is an algebra (by Section 2.3). Therefore by definition of measurable function (Proposition 3.1(b)), g is measurable.

Next, suppose g is measurable. We have that f equals g restricted to D and so by Proposition 3.5(ii), f is measurable. ∎