Real Analysis 1, MATH 5210, Fall 2020

Homework 2, Riemann-Lebesgue Theorem

Due Friday, September 4, at 11:59 p.m.

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

RI.B. Prove Theorem 6-4: A bounded function f defined on $[a,b]$ is Riemann integrable on $[a,b]$ if and only if, given $\varepsilon > 0$, there is a partition $P(\varepsilon)$ of $[a,b]$ such that $\overline{S}(f;P(\varepsilon)) - \underline{S}(f;P(\varepsilon)) < \varepsilon$. HINT: If P and Q are partitions of $[a,b]$ with $P \subset Q$ then Q is a *refinement* of P. You may assume $\underline{S}(f;P) \leq \underline{S}(f;Q)$ and $\overline{S}(f;Q) \leq \overline{S}(f;P)$ (this is Exercise RI.G).

RI.C. Prove Theorem 6-5: Suppose f is a Riemann integrable function on $[a,b]$. If I is a number such that $\underline{S}(f;P) \leq I \leq \overline{S}(f;P)$ for every partition P of $[a,b]$, then $I = \int_a^b f$.

RI.D. Prove that if bounded function f is Riemann integrable on $[a,b]$ then there is a number I such that, given $\varepsilon > 0$, there is a partition P_ε such that, for any Riemann sum $S(f;P_\varepsilon)$ of f with respect to P_ε, $|S(f;P_\varepsilon) - I| < \varepsilon$. NOTE: The converse of this result also holds (see Exercise RI.E). Therefore this result and its converse could used as the definition of the Riemann integral of f on $[a,b]$ since since this is an if and only if result. Notice that there is no explicit mention of suprema nor infima here, only function values (function values are in in the Riemann sum). This is the approach to Riemann integration taken in Calculus 1.

RI.G. **(a)** Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and let

$$Q = P \cup \{x'\} = \{x_0, x_1, \ldots, x_k, x', x_{k+1}, \ldots, x_n\}$$

(Q is a *one point refinement of* P). Prove that $\underline{S}(f;P) \leq \underline{S}(f;Q)$. You may assume that $A \subset B$ implies $\inf(B) \leq \inf(A)$.

(b) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and let Q be a partition of $[a,b]$ such that $P \subset Q$ (Q is called a *refinement* of P). Prove that $\underline{S}(f;P) \leq \underline{S}(f;Q)$. NOTE: A similar argument shows that $\overline{S}(f;P) \geq \overline{S}(f;Q)$.