Real Analysis 1, MATH 5210, Fall 2020

Homework 6, 2.4. Outer and Inner Approximation of Lebesgue Measurable Sets, Solutions

Due Monday, October 5, by noon

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

2.16(a). Partially complete the proof of Theorem 2.11 by showing (iii) set E is measurable if and only if for all $\varepsilon > 0$ there is a closed set F with $F \subset E$ for which $m^*(E \setminus F) < \varepsilon$. HINT: E is measurable if and only if E^c is measurable, and $E \setminus \mathcal{O}^c = \mathcal{O} \setminus E^c$. To show that E is measurable given such closed set F, consider the (easily verified) fact that $A \cap E = (A \cap F) \cup (A \cap (E \setminus F))$.

2.16(b). Complete the proof of Theorem 2.11 by showing (iv) set E is measurable if and only if there is an F_σ set F with $F \subset E$ for which $m^*(E \setminus F) = 0$. HINT: To show such a set F exists, use (iii). To show the converse, use the hint for (iii).

2.19. Let E have finite outer measure. Prove that if E is <u>not</u> measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \setminus E) > m^*(\mathcal{O}) - m^*(E).$$

NOTE: This is our first encounter with the behavior of a (Lebesgue) non-measurable set. It will get weirder.

2.20. (Lebesgue) Let E have finite outer measure. Prove that E is measurable if and only if for each open, bounded interval (a, b),

$$m^*((a, b)) = m^*((a, b) \cap E) + m^*((a, b) \cap E^c).$$

NOTE: This result implies that the Carathéodory splitting condition need not be checked for *all* sets of real numbers, but merely for open bounded intervals.