Real Analysis 1, MATH 5210, Fall 2020

Homework 8, 2.7. The Cantor Set and the Cantor Lebesgue Function, Solutions

Due Sunday, October 15, by noon

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

2.34. Prove that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero. HINT: You may assume Problem 2.8.A, which states that for any $E \subset \mathbb{R}$ we have for $k \in \mathbb{R}$ that $m^*(kE) = |k|m^*(E)$ where $kE = \{ke \mid e \in E\}$.

2.39. Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Prove that F is a closed set, $[0, 1] \setminus F$ is dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a *generalized Cantor set*.

2.41. A nonempty subset X of $\mathbb{R}$ is *perfect* provided it is closed and each neighborhood of any point of X contains infinitely many points of X. Prove that the Cantor set is perfect. HINT: The endpoints of all of the subintervals occurring in the Cantor construction belong to $\mathbf{C}$.

2.44. A subset of $\mathbb{R}$ is *nowhere dense* in $\mathbb{R}$ provided that every open set $\mathcal{O}$ has an open subset that is disjoint from A. Prove that the Cantor set is nowhere dense in $\mathbb{R}$.