Real Analysis 1, MATH 5210, Fall 2024

Homework 1, Essential Background, and the Riemann-Lebesgue Theorem

Due Saturday, August 31, at 11:59 p.m.

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. Do not discuss homework problems with others. If you have any questions, then contact me (`gardnerr@etsu.edu`). Use the same notation and terminology we used in class and given in the in-class hints. Links to the videos can also be found on my "Online Real Analysis 1, with online videos" webpage. This homework set should be done **in your own handwriting**. In future assignments, typeset solutions are acceptable (but not required).

Background 1. Watch the YouTube video Essential Background for Real Analysis 1 and pay attention to it. To get credit for this problem, after watching the video, in your own handwriting, write out the statement: "I, [state your name], have watched the YouTube video 'Essential Background for Real Analysis 1' and have paid attention to it. [Insert your signature]" Also write out any questions you have on details. I will either direct you to other resources or we can discuss your questions in class, as time permits. (10 points)

Background 2. State the theorems or definitions of the following (i.e., define the italicized terms). You may copy the definitions from my online notes on Essential Background for Real Analysis 1, which contain all of these. (5 points)

- **(a)** *Ordered field* (you don't need to define a field).
- **(b)** *Least upper bound* and *greatest lower bound* for a bounded set of real numbers.
- **(c)** *Completeness* in an ordered field and the definition of the *real numbers*.
- **(d)** State the "Epsilon Property of Sup and Inf."
- **(e)** *Cauchy sequence* of real numbers.
- **(f)** *Open set* of real numbers and *closed set* of real numbers,
- **(g)** State the "Classification of Open Sets of Real Numbers" theorem (this result is a **very big deal** for us).
- **(h)** *Compact set* of real numbers

(i) *Same cardinality* sets and *countable set*.

(j) State "Cantor's Theorem" and explain its significance.

Riemann-Lebesgue Theorem 1. Watch the YouTube video 'The Riemann-Lebesgue Theorem,' Part 1, Part 2, and Part 3 and pay attention to it. To get credit for this problem, after watching the video, in your own handwriting, write out the statement: "I, [state your name], have watched the YouTube video 'The Riemann-Lebesgue Theorem' and have paid attention to it. [Insert your signature]" Also write out any questions you have on details. I will either direct you to other resources or we can discuss your questions in class, as time permits. (10 points)

Riemann-Lebesgue Theorem 2. State the theorems or definitions of the following (i.e., define the italicized terms). You may copy the definitions from my online notes on The Riemann-Lebesgue Theorem, which contain all of these. (5 points)

(a) *Upper Riemann sum* and *lower Riemann sum*.

(b) *Riemann sum*.

(c) *Upper Riemann integral*, *lower Riemann integral*, and *Riemann integral*.

(d) *Norm* of a partition.

(e) Give a necessary and sufficient condition for Riemann integrability (this is Theorem 6-6 in the notes) and one sufficient condition for Riemann integrability (this is Theorem 6-7 in the notes).

(f) *Measure zero* set of real numbers.

(g) State the "Riemann-Lebesgue Theorem." This is one **major motivation** for studying Lebesgue integration.

(h) Give an example of a function that is not Riemann integrable.

(i) Sequence of function $\{f_n\}$ *converges uniformly* to function f on set $E \subset \mathbb{R}$.

(j) State a "convergence theorem" for Riemann integration (this is Theorem 8-3 in the notes). This is a second **major motivation** for studying Lebesgue integration.

1.34(a). Prove that if we take the Heine-Borel Theorem as an *axiom*, then we can *prove* the Axiom of Completeness. As you see in Analysis 1 (MATH 4127/5127), the Axiom of Completeness implies the Heine-Borel Theorem (see the proof of the Heine-Borel Theorem, Theorem 3-10, in my online notes for Analysis 1 [MATH 4127/5127] on Section 3.1. Topology of the Real

Numbers). Hence, this exercise shows that the Heine-Borel Theorem is equivalent to the Axiom of Completeness in the setting of $\mathbb{R}$. HINT: You may assume that M is a point of closure of set B of real numbers if and only if for all $\varepsilon > 0$, the interval $(M-\varepsilon, M+\varepsilon)$ contains an element of set B. (5 points)