# Real Analysis 1, MATH 5210, Fall 2024

## Homework 3, Section 2.1. Introduction, Solutions

Due Saturday, September 14, at 11:59 p.m.

**Write in complete sentences!!!** *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. Do not discuss homework problems with others. If you have any specific questions, then contact me (`gardnerr@etsu.edu`).

**2.3.** Let $m'$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0,\infty]$. Assume $m'$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m'(\cup_{k=1}^{\infty} E_k) \le \sum_{k=1}^{\infty} m'(E_k)$.

**2.1.B.** Let $m'$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0,\infty]$. Assume $m'$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$ where $E_k \supset E_{k+1}$ for all $k \in \mathbb{N}$. Suppose $m'(E_1) < \infty$. Prove that $m'(\cap_{k=1}^{\infty} E_k) = \lim_{k\to\infty} m'(E_k)$. This is called *continuity with respect to decreasing sequences*. HINT: You may assume Problem 2.1.A: For $\{E_k\}_{k=1}^{\infty}$ a countable collection of sets in $\mathcal{A}$ where $E_k \subset E_{k+1}$ for all $k \in \mathbb{N}$, we have $m'(\cup_{k=1}^{\infty} E_k) = \lim_{k\to\infty} m'(E_k)$. This is called *continuity with respect to increasing sequences.*

**2.1.C.** Let $S$ be a countable set, and consider the power set $\mathcal{P}(S)$ as a $\sigma$-algebra $\mathcal{A}$ of subsets of set $S$. Let $m'$ be a set function defined for all sets in the $\sigma$-algebra $\mathcal{A}$, with values in $[0,1]$. Assume $m'$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $p = m'$ such that any two subsets of $S$ of the same cardinality have the same measure. Prove that $\mathcal{A} = \mathcal{P}(S)$ cannot be a sample space with probability measure $p = m'$ by showing no event has probability 1: $p(E) = m'(E) \ne 1$ for all $E \in \mathcal{P}(S) = \mathcal{A}$. This shows that there cannot be a uniform probability function on a countable set.