Real Analysis 1, MATH 5210, Fall 2024

Homework 5, Section 2.3. The σ-Algebra of Lebesgue Measurable Sets

Due Saturday, October 12, at 11:59 p.m.

Write in complete sentences and paragraphs!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. Use the notation and techniques described in the in-class hints. Do not discuss homework problems with others. If you have any questions, then contact me (`gardnerr@etsu.edu`).

(2.11 and 2.12) 2.11. Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains all intervals of the form (a, ∞) where $a \in \mathbb{R}$, then it contains all intervals. HINT: There are eight types of intervals: (1) (a, ∞), (2) $[a, \infty)$, (3) $(-\infty, b)$, (4) $(-\infty, b]$, (5) (a, b), (6) $[a, b)$, (7) $(a, b]$, and (8) $[a, b]$.

2.12. Prove that every interval is a Borel set. HINT: Use Problem 2.11.

2.15. Prove that if E is a measurable set with finite outer measure and $\varepsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has outer measure at most ε. HINT: Cut set E into bounded pieces and a single unbounded piece which is small in measure.

2.3.A. The symmetric difference of sets A and B is $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that if A is measurable and $m^*(A \Delta B) = 0$ then B is measurable and $m^*(A) = m^*(B)$. HINT: Use the fact that the measurable sets form a σ-algebra (by Theorem 2.9) to write B in terms of measurable sets using unions, intersections, and complements. Use the countable additivity of m^* on measurable sets (Proposition 2.13).