Real Analysis 1, MATH 5210, Fall 2024

Homework 7, Section 2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma, 2.6. Nonmeasurable Sets

Due Saturday, November 2, at 11:59 p.m.

Write in complete sentences and paragraphs!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. Use the notation and techniques described in the in-class hints. Do not discuss homework problems with others. If you have any questions, then contact me (`gardnerr@etsu.edu`).

2.27. Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbb{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive (that is, for all infinite collections of sets $\{E_k\}_{k=1}^{\infty}$, where the E_k are pairwise disjoint, we have $m'(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m'(E_k)$), and such that $m'(\varnothing) = 0$.

(i) Prove that m' is finitely additive, monotone, countably monotone, and possesses the excision property.

HINT: "Countably monotone" means that for all $E \in \mathcal{M}'$ such that $E \subset \cup_{k=1}^{\infty} E_k$ for $E_k \in \mathcal{M}'$, $1 \le k < \infty$, we have $m'(E) \le \sum_{k=1}^{\infty} m'(E_k)$. This is defined in Section 17.3. The Carathéodory Measure Induced by an Outer Measure.

2.30. Prove that any choice set for the rational equivalence relation $\sim$ on a set A of positive outer measure must be uncountably infinite. That is, any set C given by the Axiom of Choice by choosing a representative from each equivalence class of $\sim$ must be uncountable.

Problem 2.6.B. Show that if A is any set with $m^*(A) > 0$, then there is a nonmeasurable set $E \subset A$. HINT: Find an interval $[k, k+1)$ such that $A_k = A \cap [k, k+1)$ has positive outer measure. Partition A_k using P_i for $1 \le i < \infty$ and show some set in this partition is not measurable.