Real Analysis 2, MATH 5220

Homework 1, Section 5.2

Due Tuesday January 27, 2015 at 2:15

5.6. Let $\{f_n\} \to f$ in measure on E and let g be a measurable function on E that is finite a.e. on E. Then $\{f_n\} \to g$ in measure on E if and only if $f = g$ a.e. on E. NOTE: This result shows that the limit in measure of a sequence of functions in unique (up to "a.e."). HINT: The proof of $f = g$ a.e. implies $\{f_n\} \to g$ in measure is easy. To show that $f = g$ a.e. if convergence in measure holds, do by contradiction. Suppose $E_0 = \{x \in E \mid f(x) \neq g(x)\}$ satisfies $m(E_0) > 0$. Write E_0 as a union of an ascending sequence of sets and use Continuity of Measure to find a set of positive measure on which $|f(x) - g(x)| > K$ for some fixed $K > 0$. Use this set to show that either $\{f_n\}$ does not converge to f in measure or that $\{f_n\}$ does not converge to g in measure.

5.8a. Fatou's Lemma holds for convergence in measure: Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ in measure, then

$$\int_E f \leq \liminf \left(\int_E f_n \right).$$

HINT: Consider a subsequence of $\{\int_E f_n\}_{n=1}^{\infty}$ which converges to $\liminf(\int_E f_n)$. Apply Theorem 5.4 to the subsequence and consider the associated integrals.

5.8b. The Monotone Convergence Theorem holds for convergence in measure: Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ in measure on E, then

$$\lim_{n\to\infty} \left(\int_E f_n \right) = \int_e \left(\lim_{n\to\infty} f_n \right) = \int_E f.$$