Real Analysis 2, MATH 5220

Homework 4, Section 7.2

Due Thursday February 26, 2015 at 2:15

7.12. For $1 \leq p < \infty$ and a sequence $a = (a_1, a_2, \ldots) \in \ell^p$, define T_a to be the function on the interval $[1, \infty)$ that takes the value a_k on $[k, k+1)$, for $k = 1, 2, \ldots$. In parts (a) and (b) we give a proof of Hölder's Inequality for ℓ^p using Hölder's Inequality in $L^p[1, \infty)$.

(a) Show that $T_a \in L^p[1, \infty)$ and $\|a\|_p = \|T_a\|_p$.

(b) Prove a Hölder's Inequality for ℓ^p.

7.12 (cont.) In parts (c) and (d) we give a proof of Minkowski's Inequality for ℓ^p using Minkowski's Inequality in $L^p[1, \infty)$.

(c) For $a \in \ell^p$, define a^*, show $a^* \in \ell^q$, and that $\displaystyle\sum_{k=1}^{\infty} a_k a_k^* = \|a\|_p$.

(d) Prove a Minkowski Inequality for ℓ^p. NOTE: The Minkowski Inequality allows us to conclude that ℓ^p is a normed linear space.

7.18. Assume $m(E) < \infty$. For $f \in L^\infty(E)$, show that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$. HINT: First show that $\limsup\limits_{p \to \infty} \|f\|_p \leq \|f\|_\infty$. Second, let $\varepsilon > 0$ and define $A = \{x \in E \mid |f| \geq \|f\|_\infty - \varepsilon\}$. Show that $\liminf\limits_{p \to \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$.

7.7b. (Bonus) Let $E = (0, \infty)$ and define $f(x) = \dfrac{x^{-1/2}}{1 + |\ln x|}$. Show that $f \in L^p(0, \infty)$ if and only if $p = 2$. NOTE: There is a typographical error on page 143 in the second example. The function that is given in the book can be shown to be in $L^p(1, \infty)$ for $2 \leq p \leq \infty$, and so the claim in the example is inaccurate.