Real Analysis 1, MATH 5210, Spring 2025

Homework 8, Hong/Wang/Gardner Section 5.3. The Space L^2, Section 5.4. Projections and Hilbert Space Isomorphisms

Due Saturday, April 5, at 11:59 p.m.

Write in complete sentences and paragraphs!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses. Use the notation and techniques described in the in-class hints. Do not discuss homework problems with others. If you have any questions, then contact me (`gardnerr@etsu.edu`).

5.3.5. As set D is said to be *dense* in ℓ^2 if the topological closure of D is ℓ^2. That is, if $\mathbf{x} \in \ell^2$ then for all $\varepsilon > 0$ there exists $\mathbf{d} \in D$ such that $\|\mathbf{x} - \mathbf{d}\|_2 < \varepsilon$. Find a countable dense subset of ℓ^2.

5.4.6. Prove that $R = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots\} \subset \ell^2$ is a (topologically) closed set and a bounded set, but not a compact set. NOTE: Recall that the Heine-Borel Theorem (Theorem 1.4.11) states that a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. The example given here shows that the familiar Heine-Borel Theorem does not hold in all metric spaces. Also notice that R is an infinite bounded set with no limit points, indicating that the Bolzano-Weierstrass Theorem does not hold in ℓ^2.