Real Analysis 2, MATH 5220

Homework 3, Sections 5.1 and 5.2 (Revised)

Due Friday February 15, 2013 at 2:30

5.3. Let the sequences of function $\{f_n\}$ and $\{g_n\}$ be tight over E. Then for any $\alpha, \beta \in \mathbb{R}$, the sequence $\{\alpha f_n + \beta g_n\}$ is tight over E. HINT: Take your lead from Problem 5.2 which we did in class. Break into two parts: The sequence $\{f_n + g_n\}$ and the sequence $\{\alpha f_n\}$.

5.4. Bonus. Let $\{f_n\}$ be a sequence of measurable functions on E. Then the sequence $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\epsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n,

$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \epsilon.$$

HINT: (1) $\epsilon/\delta/A/E_0$ condition implies uniformly integrable. This is easy. (2) $\epsilon/\delta/A/E_0$ implies tight. Let $A = E \setminus E_0$. (3) uniformly integrable and tight implies $\epsilon/\delta/A/E_0$. Use an $\epsilon/2$ argument with half coming from uniformly integrable and half from tight. The A here is not the same A as given by uniformly integrable. Also,

$$\int_A |f_n| = \int A \cap (E \setminus E_0)|f_n| + \int_{A \cap E_o} |f_n|.$$

5.6. Let $\{f_n\} \to f$ in measure on E and let g be a measurable function on E that is finite a.e. on E. Then $\{f_n\} \to g$ in measure on E if and only if $f = g$ a.e. on E. NOTE: This result shows that the limit in measure of a sequence of functions in unique (up to "a.e."). HINT: The proof of $f = g$ a.e. implies $\{f_n\} \to g$ in measure is easy. To show that $f = g$ a.e. if convergence in measure holds, do by contradiction. Suppose $E_0 = \{x \in E \mid f(x) \neq g(x)\}$ satisfies $m(E_0) > 0$. Write E_0 as a union of an ascending sequence of sets and use Continuity of Measure to find a set of positive measure on which $|f(x) - g(x)| > K$ for some fixed $K > 0$. Use this set to show that either $\{f_n\}$ does not converge to f in measure or that $\{f_n\}$ does not converge to g in measure.

5.8a. Fatou's Lemma holds for convergence in measure: Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ in measure, then

$$\int_E f \leq \liminf \left(\int_E f_n \right).$$