# Real Analysis 2, MATH 5220

## Homework 5, Section 7.2

Due Friday March 8, 2013 at 2:30

**7.12.** For $1 \leq p < \infty$ and a sequence $a = (a_1, a_2, \ldots) \in \ell^p$, define $T_a$ to be the function on the interval $[1, \infty)$ that takes the value $a_k$ on $[k, k+1)$, for $k = 1, 2, \ldots$.

- **(a)** Show that $T_a \in L^p[1, \infty)$ and $\|a\|_p = \|T_a\|_p$.
- **(b)** Prove a Hölder's Inequality for $\ell^p$.
- **(c)** For $a \in \ell^p$, define $a^*$, show $a^* \in \ell^q$, and that $\sum_{k=1}^{\infty} a_k a_k^* = \|a\|_p$.
- **(d)** Prove a Minkowski Inequality for $\ell^p$. NOTE: The Minkowski Inequality allows us to conclude that $\ell^p$ is a normed linear space.

**7.13.** Prove that if $f$ is a bounded function on $E$ that belongs to $L^{p_1}(E)$, then $f$ belongs to $L^{p_2}(E)$ for any $p_2 > p_1$. HINT: Define $E_1 = \{x \in E \mid |f(x)| \leq 1\}$ and $E_\infty = E \setminus E_1$.

**7.18.** Assume $m(E) < \infty$. For $f \in L^\infty(E)$, show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. HINT: First show that $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$. Second, let $\epsilon > 0$ and define $A = \{x \in E \mid |f| \geq \|f\|_\infty - \epsilon\}$. Show that $\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty - \epsilon$.

**7.7b. (Bonus)** Let $E = (0, \infty)$ and define $f(x) = \dfrac{x^{-1/2}}{1 + |\ln x|}$. Show that $f \in L^p(0, \infty)$ if and only if $p = 2$. NOTE: There is a typographical error on page 143 in the second example. The function that is given in the book can be shown to be in $L^p(1, \infty)$ for $2 \leq p \leq \infty$, and so the claim in the example is inaccurate.