Real Analysis 2, MATH 5220

Homework 7, Section 17.1

17.1. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$, define $\mu(E) = \int_E f$. Show that μ is a measure on the σ-algebra of Lebesgue measurable sets. HINT: Show μ is nonnegative ("clearly"), $\mu(\varnothing) = 0$, and μ is countably additive. Notice that for $f \equiv 1$, μ is Lebesgue measure.

17.5. Let $(X, \mathcal{M}, \mu)$ be a measure space. the symmetric difference, $E_1 \triangle E_2$, of two subsets E_1 and E_2 of X is defined as $E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

(i) Show that if $E_1, E_2 \in \mathcal{M}$ and $\mu(E_1 \triangle E_2) = 0$, then $\mu(E_1) = \mu(E_2)$.

(ii) Show that if μ is complete and $E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ if $\mu(E_1 \triangle E_2) = 0$.

HINT: For (ii), we have $E_1 \setminus E_1 \setminus E_2) = E_1 \cap E_2$ and $E_2 = (E_2 \setminus E_1) \cup (E_1 \cap E_2)$.

17.7. Let $(X, \mathcal{M})$ be a measurable space.

(i) If μ and ν are measures defined on $\mathcal{M}$, then the set function λ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure, denoted $\lambda = \mu + \nu$.

(ii) [**Bonus**] If μ and ν are measures on $\mathcal{M}$ and $\mu \geq \nu$ then there is a measure λ on $\mathcal{M}$ for which $\mu = \nu + \lambda$. HINT: Define

$$\lambda(E) = \begin{cases} \mu(E) - \nu(E) & \text{if } \nu(E) < \infty \\ \sup\{\mu(F) - \nu(F)\} & \text{if } \nu(E) = \infty \end{cases}$$

where the supremum is taken over all $F \in \mathcal{M}$, $F \subset E$, and $\nu(E) < \infty$. Show (1) $\mu = \nu + \lambda$ on $\mathcal{M}$, (2) λ is finitely additive, and (3) λ is countably additive.

(iii) [**Bonus**] If ν is σ-finite, then the measure λ on (ii) is unique. HINT: This is easy for finite ν-measure sets. Use the σ-finiteness of ν to show this for arbitrary $E \in \mathcal{M}$.

(iv) [**Bonus**] In general, the measure λ of (ii) need not be unique, but there is always a smallest such λ. HINT: Use finite ν-measure sets to compare any two such λ and the fact that λ of (ii) is defined in terms of a least upper bound.