Real Analysis 1, "Test 2" Study Guide

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1.4 Algebras of Sets. Algebra of sets, algebra generated by a collection of sets, σ-algebra of sets, σ-algebra generated by a collection of sets.

Axiom of Choice. Ernst Zermelo, Axiom of Choice, well orderings (binary relation, partial ordering, total ordering), Well Ordering-Principle.

Chapter 2. Real Number Systems. Completeness Axiom, Complete Ordered Field, what does an open set of real numbers "look like"?, Lindelöf Principle, Borel sets, F_σ/G_δ, $F_{\sigma\delta}/G_{\delta\sigma}$, $f : \mathbb{R} \to \mathbb{R}$ has an F_σ set of discontinuities, how man Borel sets are there?

3.1 Introduction to Lebesgue Measure. The four desired properties of a measure, monotonicity, translation invariance, countable subadditivity.

3.2 Outer Measure. Outer measure, outer measure of an interval (Theorem 3.1), outer measure is countably subadditive (Theorem 2), countable sets are measure zero (Corollary 3), approximation of a set with a G_δ set (Theorem 5), outer measure is translation invariant (Exercise 3.7).

3.3 Measurable Sets and Lebesgue Measure. (Lebesgue) measurable, outer measure zero implies measurable (Lemma 6), the measurable sets form an algebra of sets (Lemma 7 and Corollary 8), the measurable sets form a σ-algebra, (a, ∞) is measurable (Lemma 11), the Borel sets are measurable (Theorem 12), definition of Lebesgue measure, Lebesgue measure is countably additive (Theorem 13), approximation of measurable sets with open/closed/F_σ/G_δ sets (Theorem 15), Cantor set (it's measure, cardinality and use to determine how many measurable sets).

3.4 A Nonmeasurable Set. Construction of the nonmeasurable set P and the idea behind how it is shown to be nonmeasurable.

The Banach-Tarski Paradox. Georg Cantor, shifting to infinity, the Hilbert Hotel, Hausdorff Paradox.

4.1 The Riemann Integral. Step functions, upper and lower Riemann integrals in terms of suprema and infema of step functions, limits of sequences of Riemann integrable functions and uniform convergence, necessary and sufficient conditions for a bounded function to be Riemann integrable.

4.2 The Lebesgue Integral of a Bounded Function over a Set of Finite Measure. Characteristic function, simple function, canonical representation, linearity of integrals of simple functions, relationship between sup and inf of integrals of step functions (Theorem 3), (Lebesgue) integral of a bounded function on a set of finite measure, agreement of Riemann and Lebesgue integrals (Theorem 4), linearity and inequalities and bounds of integrals (Theorem 5), Bounded Convergence Theorem.

4.3 The Integral of a Nonnegative Function. Definition of (Lebesgue) integral of a nonnegative function, linearity and inequalities of integrals (Theorem 8), Fatou's Lemma and example showing the inequality can be strict, Monotone Convergence Theorem, definition of *integrable*, continuity of integrals (Theorem 14).

4.4 The General Lebesgue Integral. Positive and negative part, *integrable* measurable functions, linearity and inequalities for integrals (Theorem 15), Lebesgue Dominated Convergence Theorem.

Covered in class but not included on Test 2:

Chapter 5 of Hong, Wang, Gardner. Vector Spaces, Hilbert Spaces, and the L^2 Space. Vector spaces over scalar fields, Fundamental Theorem of Finite Dimensional Vector Spaces, Hamel basis, every vector space has a Hamel basis (Theorem 5.1.4), Schauder basis, inner product space, norm, metric, Schwarz's Inequality, Triangle Inequality, orthogonal and orthonormal sets, Bessel's Inequality, Cauchy sequences, completeness, Banach Space, Hilbert Space, ℓ^2, L^2, $\mathbb{R}^\infty$, projections, orthogonal complements, Gram-Schmidt Process and Theorem 5.4.4, separable Hilbert space, Hilbert space isomorphism, Fundamental Theorem of Infinite Dimensional Vector Spaces.