Nonmeasurable sets and the Banach-Tarski Paradox

Based largely on *The Pea and the Sun—A Mathematical Paradox*, by Leonard M. Wapner.
Wellesley, Massachusetts: A.K. Peters, Ltd., 2005.
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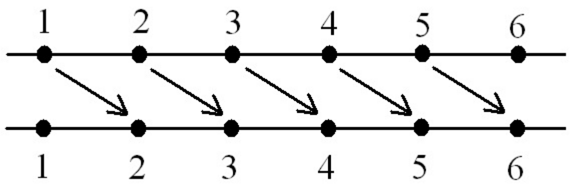
Note. Georg Cantor (1845–1918) defined an infinite set as one for which there is a one-to-one mapping from the set to a proper subset.

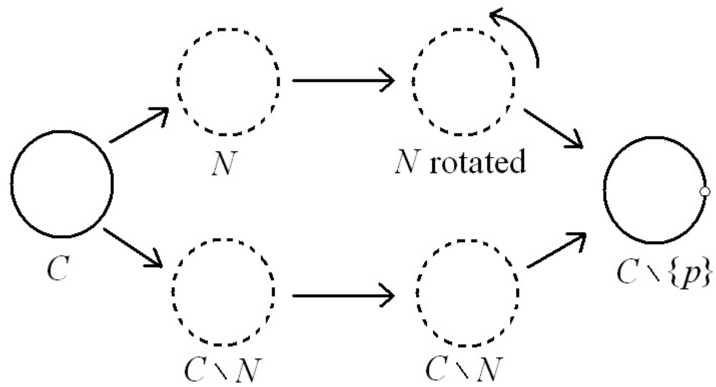
Note. We can use Cantor's definition to show that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is an infinite set. The function $f : x \to x + 1$ sends $\mathbb{N}$ to $\mathbb{N} \setminus \{1\}$. If we imagine $\mathbb{N}$ as a collection of points on the real line, then the mapping can be illustrated as:



Such a mapping is called *shifting to infinity.* It is a rigid mapping (distances between corresponding points are preserved—it is an *isometry*) and there is a point in $\mathbb{N}$ that is not in $F(\mathbb{N}) = \mathbb{N} \setminus \{1\}$. That is, 1 has been "shifted to infinity" and lost in the rigid transformation.

Note. There is a story involving a hotel with a countably infinite number of rooms. Every room is occupied. If a new person arrives, then they can be given a room if the person in room i moves to room $i + 1$. In fact, an infinite number of new people can be added by having the person in room i move to room $2i$. This is sometimes called the "Hilbert Hotel."

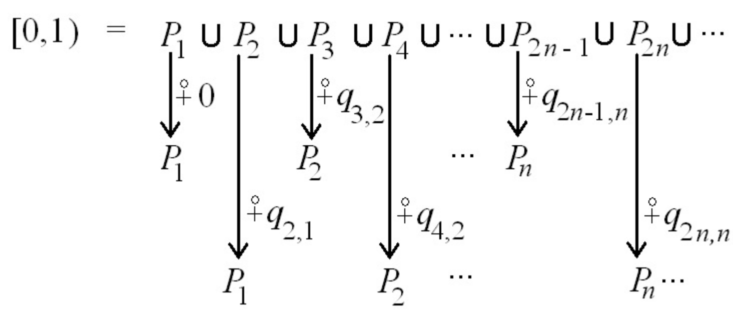
Note. Consider the unit circle in $\mathbb{R}^2$, $C = \{(x, y) \mid x^2 + y^2 = 1\}$. Create set N by taking the point $(1, 0)$, the point one unit of arc length away (in the counterclockwise direction), the point two units of arc length away (counterclockwise), and so forth. Since the circumference of the circle is irrational, the set N is infinite. Now decompose the circle into two disjoint sets $C = N \cup (C \setminus N)$. Now rigidly rotate N through one radian. This maps the nth point of N to the $(n + 1)$st point of N. Leave $C \setminus N$ fixed. We have then taken C, broken it into two disjoint pieces, rigidly rearranged the pieces, and produced a copy of C with a missing point (again, the point has been "shifted to infinity").



Of course, we could shift any finite number of points to infinity in this way. We can also perform similar tricks on the unit disk and "shift to infinity" line segments from the center to the edge of the disk. These processes can be reversed to "shift from infinity" to produce extra points.

Note. We now perform a magic trick! We will break the interval $[0, 1)$ into pieces, rigidly rearrange the pieces, and create two copies of $[0, 1)$.

Recall. We have partitioned $[0, 1)$ into a countable number of disjoint sets $\{P_i\}_{i=1}^{\infty}$ such that any P_i can be rigidly translated into any P_j by adding an appropriate rational number (modulo 1): $P_i = P_j \overset{\circ}{+} q_{ij}$ for some $q_{ij} \in \mathbb{Q} \cap [0, 1)$ where $\overset{\circ}{+}$ is as defined in Royden's *Real Analysis*. We have seen that $\cup_{i=1}^{\infty} P_i = [0, 1)$. This construction was originally done by Giuseppe Vitali (1874–1932) in *Sul problema della misura dei gruppi di punti di una retta* Bologna: Tip, Gamberini e Parmeggiani (1905). The set is often called the *Vitali set*. We now show that we can also have $\cup_{i=1}^{\infty} P_i = 2 \times [0, 1)$.



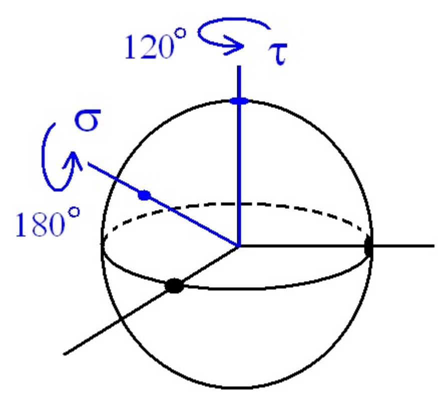
We now have two copies of each P_i, and can union them to produce two copies of $[0, 1)$. We can translate one copy to produce $[0, 2)$ if you like. We can modify the mapping of even and odd indexed P_is to produce any number of copies of $[0, 1)$ (including an infinite number of copies), or an interval of any length (including infinite, or all of $\mathbb{R}$).

Note. Informally, we can wrap $[0,1)$ around the unit circle (this requires a bit of stretching), and use the above argument to create 2 circles from 1... or n circles from 1, or an infinite number of circles from 1!

Note. The Banach-Tarski Theorem states [Wapner, p. 143]: "A solid ball may be separated into a finite number of pieces and reassembled in such a way as to create two solid balls, each identical in shape and volume to the original." Wapner also comments (p. 45): "The Banach-Tarski Paradox does not hold in the plane; a space of three or more dimensions is required." We have seen similar paradoxes for an interval and a disk, but our examples required an *infinite* number of pieces.

Note. Polish mathematician Stefan Banach (1892–1945), of "Banach space" fame, and Alfred Tarski (1902–1983) published "On the Decomposition of Sets of Points in Respectively Congruent Parts" (in French in *Fundamenta Mathematicae* **6**) in 1924. Their work was heavily dependent on ealier work of Vitali and Hausdorff.

Note. A nice, not-too-technical description of the proof of the Banach-Tarski Paradox is given in Wapner's book. We give a vague description of it here. First, we consider two rotations of the unit sphere $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$. One is a 180° rotation about the line $x = z$ (call this σ), the other a 120° rotation about the z-axis (call this τ).



By taking combinations of σ and τ, we can generate an infinite (countable) group G which maps S onto itself. We then partition G into three pieces, G_1, G_2, and G_3 (details on page 147 of Wapner).

This is done is such a way that the following hold:

$$\begin{aligned}\tau G_1 &= G_2\\ \tau^2 G_1 &= G_3\\ \sigma G_1 &= G_2 \cup G_3\end{aligned}$$

Since each element of G is a type of rotation, then each such element has two fixed points associated with it (the poles of the rotation). Denote the collection of all such poles as P. The group will partition the remainder of the points, $S \setminus P$, into an uncountable number of orbits. Next, use the Axiom of Choice to choose a representative of each of these orbits and call the resulting set C. Set C is similar to the Vitali set from interval $[0, 1)$. We now apply all of the permutations from the three pieces of G to create a partition of the sphere. Apply the rotations of G_i to C to create set K_i for $i = 1, 2, 3$. We have now partitioned the unit sphere into four pieces: $S = K_1 \cup K_2 \cup K_3 \cup P$. Because of the relationships between the G_is, we have the following relationships between the K_is:

$$\begin{aligned}\tau K_1 &= K_2\\ \tau^2 K_1 &= K_3\\ \sigma K_1 &= K_2 \cup K_3\end{aligned}$$

Note. That is, K_1 can be rigidly rotated to give K_2 (by τ), K_3 (by τ^2), or $K_2 \cup K_3$ (by σ). When one set can be rigidly rotated to give another, we say the sets are *congruent*. So we have

$$K_1 \cong K_2 \cong K_3 \cong K_2 \cup K_3.$$

This is called the *Hausdorff Paradox*. That is, we have broken the sphere into three congruent pieces, K_1, K_2, and K_3, indicating that each piece is one-third of the sphere (actually, $S \setminus P$ has been partitioned, but remember that P is a "small" countable set). However, when we union two of these disjoint pieces, K_2 and K_3, we would expect to get two-thirds of the sphere, but instead get something which is again congruent to one-third of the sphere. Felix Hausdorff (1868–1942) presented this in his *Grundzüge der Mengenlehre* in 1914.

Note. Now here's the real trick! We use $K_2 \cup K_3$ as a "cutting template." We place $K_2 \cup K_3$ on K_1 (which can be done since they are congruent) and cut K_1 into two pieces—one piece congruent to K_2 and one piece congruent to K_3. Next, do the same to K_2 and K_3. We then have three copies of K_2 and three copies of K_3. One K_2 can be rotated to give a K_1 and one K_3 can be rotated to give another K_1. So, through partition and rotation, we have produced two copies each of K_1, K_2, and K_3. We now have produced from the unit sphere the following:

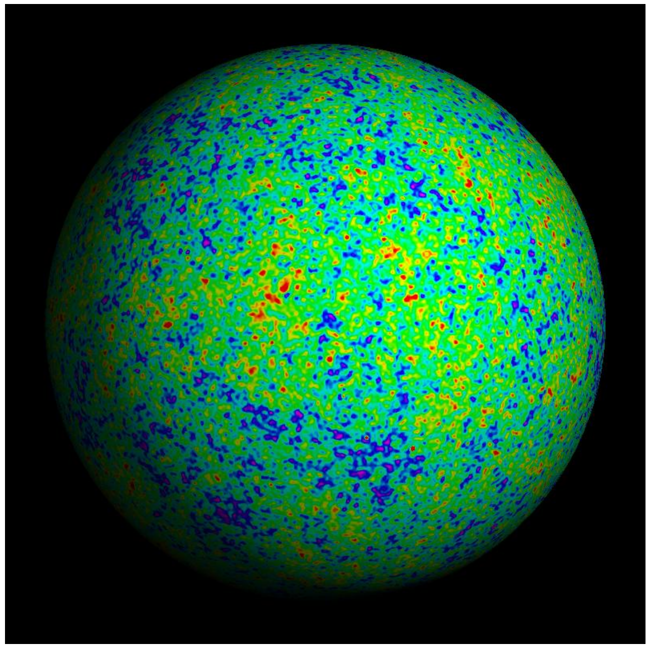
$$S = 2 \times K_1 \cup 2 \times K_2 \cup 2 \times K_3 \cup P.$$

We can produce one copy of S as $S = K_1 \cup K_2 \cup K_3 \cup P = (S \setminus P) \cup P$. Next we union the second copies of the K_is to get $S \setminus P = K_1 \cup K_2 \cup K_3$. We can then perform the "shifting from infinity" trick to plug the holes in $S \setminus P$ to produce a second, complete copy of S. Hence, we have produced two spheres from one!

Note. The above argument dealt with the two-dimensional surface of a sphere, and not with a solid ball. This part can be easily revised by taking every point on the sphere and adding an open line segment from that point to the center of the sphere. This can be done to each of the sets above as well. The same construction as above can be used, with the center of the original sphere going into one copy, and the center of the second taken care of by shifting from infinity.

Note. You probably noticed that the "cutting template" trick which was used to create a second copy of the K_is, could be used over and over to ultimately produce as many copies of the sphere as you want from the original one. Things, it turns out, are even a little stranger! There is another version of the Banach-Tarski Paradox which states that for any two bounded three-dimensional sets with nonempty interiors, one can be decomposed into a finite number of pieces and the pieces rigidly rearranged (rotated or translated) to produce the other. This implies, among other offensive things, that a sphere the size of a pea can be decomposed and rearranged to produce a sphere the size of the Sun. That's why the Banach-Tarski Paradox is sometimes called the "Pea and the Sun Paradox."

Note. How should we try to visualize this? What do the pieces look like? Several websites have images in which the sphere is cut into small, solid pieces. We know the pieces must be rather exotic, or else they would be measurable and the paradox would not appear (because of additivity). I imagine something like this:



You might think of the sets K_1, K_2, and K_3 as being the yellow, green, and blue points, respectively. We know that the sets are rather "mixed together," maybe like the rationals and irrationals in $\mathbb{R}$. There are a few more points of different colors here, which we might think of as points in set P. This image (from CMB Map Lab, from the webpage of Clem Pryke of the University of Minnesota Department of Physics) is actually a map of the cosmic microwave background.

Note. The construction presented here is not the easiest in terms of numbers of pieces. In 1947, R.M. Robinson (1911–1995) showed that the construction could be accomplished using only five pieces (and could not be accomplished using fewer pieces). Two of the pieces form one new sphere and the other three create a second sphere ("On the Decomposition of Spheres," *Fundamenta Mathematicae*, **34** (1947), 246–260).

References

1. Royden, H.R. *Real Analysis*, 3rd Edition. Prentice Hall, 1988.

2. Wagon, S. *The Banach-Tarski Paradox*. New York: Cambridge University Press, 1985. (This book is highly technical.)

3. Wapner, L.M. *The Pea and the Sun—A Mathematical Paradox*. A.K. Peters, Ltd., 2005.

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