Chapter 1. Open Sets, Closed Sets, and Borel Sets

Section 1.4. Borel Sets

Note. Recall that a set of real numbers is *open* if and only if it is a countable disjoint union of open intervals. Also recall that:

1. a countable union of open sets is open, and
2. a countable intersection of closed sets is closed.

These two properties are the main motivation for studying the following.

Definition. A collection $\mathcal{A}$ of subsets of a set X is an *algebra* (or *Boolean algebra*) of sets if:

1. $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$.
2. $A \in \mathcal{A}$ implies $\tilde{A} = X \sim A \in \mathcal{A}$ ($\tilde{A}$ is the complement of A).
3. $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$ (this follows from (1) and (2) by DeMorgan's Laws).

We also require that $\varnothing, X \in \mathcal{A}$. (This last condition, absent in previous editions of Royden, insures that an algebra is nonempty.)

Example. $\mathcal{A} = \{\varnothing, \mathbb{N}, \text{evens}, \text{odds}\}$ is an algebra on $\mathbb{N}$.

Note. By induction, (1) and (3) hold for any finite collection of elements of $\mathcal{A}$.

Theorem 1.4.A. Given any collection $\mathcal{C}$ of subsets of X, there exists a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$. That is, if $\mathcal{B}$ is any algebra containing $\mathcal{C}$, then $\mathcal{B}$ contains $\mathcal{A}$.

Definition. The smallest algebra containing $\mathcal{C}$, a collection of subsets of a set X, is called the *algebra generated by* $\mathcal{C}$.

Definition. An algebra $\mathcal{A}$ of sets is a *σ-algebra* (or a *Borel field*) if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$.

Example. Let $X = \mathbb{R}$ and $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is finite or } \tilde{A} \text{ is finite}\}$. Then $\mathcal{A}$ is an algebra but not a σ-algebra (since $\mathbb{N} = \cup\{n\}$ but $\mathbb{N} \notin \mathcal{A}$).

Proposition 1.13. Let $\mathcal{C}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{C}$ is a σ-algebra that contains $\mathcal{C}$. Moreover, it is the smallest σ-algebra of subsets of X that contain $\mathcal{C}$ in the sense that if $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$, then $\mathcal{A} \subset \mathcal{B}$.

Definition. The σ-algebra of Proposition 1.13 is the σ-algebra *generated* by $\mathcal{C}$.

Recall. A countable union of closed sets of real numbers need not be closed:

$$\cup_{n=1}^{\infty}[0+1/n, 2-1/n] = (0,2).$$

In fact, a countable union of closed sets may be neither open nor closed: $\cup_{i=1}^{\infty}\{r_i\} = \mathbb{Q}$ where the rationals are enumerated as $\mathbb{Q} = \{r_i \mid i \in \mathbb{N}\}$. We are interested in describing (or at least naming) the sets we get from countable unions, intersections, and complements of open sets. More specifically, we are interested in the "Borel sets."

Definition. The collection $\mathcal{B}$ of *Borel sets* is the smallest σ-algebra that contains all open sets of real numbers.

Note. How many Borel sets are there: $|\mathcal{B}| =$? According to Corollary 4.5.3 of Inder Rana's *An Introduction to Measure and Integration* (2nd Edition, AMS Graduate Studies in Mathematics, Volume 45, 2002), $|\mathcal{B}| = c = |\mathbb{R}|$ ($= \aleph_1$ if you buy the Continuum Hypothesis). This is bad (why?).

Note. What do Borel sets "look like"? We can describe some of them.

Definition. A set which is a countable union of closed sets is an F_σ *set.* A set which is a countable intersection of open sets is a G_δ *set.*

Note. According to Wikipedia (hmm...), "F" is for *ferme* (French for "closed") and σ for *somme* (French for "sum" or "union"). "G" is for *gebiet* (German for "neighborhood") and δ for *durchschnitt* (German for "intersection").

Note. A countable set is F_σ since it is a countable union of the singletons which compose it. Of course closed sets are F_σ. Since a countable collection of countable sets is countable, a countable union of F_σ sets is again F_σ. Every open interval is F_σ:

$$(a, b) = \cup_{n=1}^{\infty}[a + 1/n, b - 1/n]$$

(a and b could be $\pm\infty$), and hence every open set is F_σ (this is Problem 1.37).

Notice. The complement of an F_σ set is a G_δ set (and conversely).

Theorem 1.4.B. Young's Theorem. (Problem 1.56)
Let f be a real valued function defined on all of $\mathbb{R}$. The set of points at which f is continuous is a G_δ set.

Note. The converse of Young's Theorem also holds:
Theorem. (From *Real Functions* by Hahn, and *Counterexamples in Analysis* by Gelbrum and Olmstead.) If $A \subset \mathbb{R}$ is a G_δ set, then there exists $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at each point of A and discontinuous at each point of $\mathbb{R} \setminus A$.

Note. With δ for intersection and σ for union, we can construct (for example) a countable intersection of F_σ sets, denoted as an $F_{\sigma\delta}$ set. Similarly, we can discuss $F_{\sigma\delta\sigma}$ sets or $G_{\delta\sigma}$ and $G_{\delta\sigma\delta}$ sets. These classes of sets are subsets of the collection of Borel sets, but not every Borel set belongs to one of these classes.

Theorem 1.4.C. (Problem 1.57.)
Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Then the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets (i.e., is an $F_{\sigma\delta}$ set).

Note. The Borel sets, and σ-algebras of sets in general, play a huge role in probability theory. In Intermediate Probability and Statistics (not a formal ETSU class) based on DeGroot and Schervish's *Probability and Statistics* 4th Edition (Pearson, 2012), three conditions are put on the set of events which, together, imply that the set of events form a σ-algebra of sets. See my online notes for this class on Section 1.4. Set Theory. The notes for this class on Section 1.5. The Definition of Probability give several properties of probability that will overlap with properties of Lebesgue measure. My online notes for Mathematical Statistics 1 (STAT 4047/5047) on Section 1.3. The Probability Set Function also addresses the fact that the events form a σ-algebra or, in the terminology of Hogg, McKean, and Craig in their *Introduction to Mathematical Statistics*, 8th Edition (Pearson, 2019), a "σ-field." This means that sample spaces form σ-algebras and these are the sets on which a probability function is defined. For a full-blown version of a theoretical probability class, see my online notes (in preparation as of this date, fall 2022) on Measure Theory Based Probability (not a formal ETSU class). Such a class requires an intense background in analysis. Prerequisites for this are Real Analysis 1 (MATH 5210; for the introduction to measure theory and Lebesgue integration), Real Analysis 2 (MATH 5220; for abstract measure theory and integration), and Functional Analysis (for more exposure to linear operators, topology, Banach spaces, and Hilbert spaces). An outline of a year-long functional analysis class is given in my online notes (in preparation) on Functional Analysis. A one-semester "miniature" functional analysis class is ETSU's Fundamentals of Functional Analysis (MATH 5740), for which I have a complete set of notes online.

Note. As an example of the necessity of measure theory for the application of probability, consider the following situation:

> Consider a uniform probability distribution on the interval $[0, 1]$. Choose a number at random based on this distribution. What is the probability that the number is rational?

The correct answer is "the measure of the set of rationals in the interval $[0, 1]$." Any guesses as to what this measure is?

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