Section 11.4. Continuous Mappings Between Topological Spaces

Note. In this section, we deal with continuous functions and some of their properties.

Note. The following definition is a parallel to the standard \( \varepsilon-\delta \) definition of continuity where the “for all \( \varepsilon > 0 \)” has been replaced with “for any neighborhood” and “there exists \( \delta > 0 \)” has been replaced with “there exists a neighborhood.”

Definition. For topological spaces \((X, T)\) and \((Y, S)\), a mapping \(f : X \to Y\) is continuous at a point \(x_0 \in X\) provided that for any neighborhood \(O\) of \(f(x_0)\), there exists a neighborhood \(U\) of \(x_0\) for which \(f(U) \subseteq O\). \(f\) is continuous if it is continuous at each point in \(X\).

Proposition 11.10. A mapping \(f : X \to Y\) between topological spaces \((X, T)\) and \((Y, S)\) is continuous if and only if for any subset \(O \in S\), its inverse image under \(f\), \(f^{-1}(O) \in T\).

Proposition 11.11. The composition of continuous mappings between topological spaces when defined, is continuous.

Note. The straightforward proof of Proposition 11.11 is Problem 11.26.
**Definition.** Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set $X$, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then $\mathcal{T}_2$ is *weaker* than $\mathcal{T}_1$ and $\mathcal{T}_1$ is *stronger* than $\mathcal{T}_2$.

**Note.** The more open sets a topology has, the “harder” it is for sequences to converge (for example)—thus the use of “stronger” and “weaker.”

**Proposition 11.12.** Let $X$ be a nonempty set and $\mathcal{S}$ any collection of subsets of $X$ that covers $X$. The collection of subsets of $X$ consisting of intersections of finite subcollections of $\mathcal{S}$ is a base for a topology $\mathcal{T}$ for $X$. It is the weakest topology containing $\mathcal{S}$ in the sense that if $\mathcal{T}'$ is any other topology for $X$ containing $\mathcal{S}$ then $\mathcal{T} \subseteq \mathcal{T}'$.

**Note.** The proof of Proposition 11.12 is left as Problem 11.27.

**Definition.** Let $X$ be a nonempty set and consider a collection of mappings $\mathcal{F} = \{f_\alpha : X \to X_\alpha\}_{\alpha \in \Lambda}$, where each $X_\alpha$ is a topological space. The weakest topology for $X$ that contains the collection of sets

$$\mathcal{F} = \{f_\alpha^{-1}(O_\alpha) \mid f_\alpha \in \mathcal{F}, O_\alpha \text{ open in } X_\alpha\}$$

is the *weak topology* for $X$ induced by $\mathcal{F}$.

**Note.** The following result more clearly elaborates on the meaning of the weak topology.
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**Proposition 11.13.** Let $X$ be a nonempty set and $\mathcal{F} = \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$ a collection of mappings where each $X_\lambda$ is a topological space. The weak topology for $X$ induced by $\mathcal{F}$ is the topology on $X$ that has the fewest number of sets among the topologies on $X$ for which each mapping $f_\lambda : X \to X_\lambda$ is continuous.

**Definition.** A continuous mapping from a topological space $(X, T)$ to a topological space $(Y, S)$ is a homeomorphism provided it is one to one, maps $X$ onto $Y$, and has continuous inverse $f^{-1}$ from $Y$ to $X$. If there is a homeomorphism between two topological spaces, the spaces are homeomorphic.

**Note.** Homeomorphic topological spaces are indistinguishable from each other as topological spaces. So if $f : (X, T) \to (Y, S)$ is a homeomorphism then $E \subseteq X$ is open in $(X, T)$ if and only if $f(E)$ is open in $S$. However, this does not mean that the two spaces share other properties. In the next example, we show that $L^1(E)$ is homeomorphic to $L^2(E)$ for measurable set $E$.

**Example. Mazur’s Example.**

Let $E \subseteq \mathbb{R}$ be Lebesgue measurable. For $f \in L^1(E)$, define function $\Phi(f)$ on $E$ by $\Phi(f)(x) = \text{sgn}(f(x))|f(x)|^{1/2}$. Then $\Phi(f)$ clearly belongs to $L^2(E)$. We “leave as an exercise” (a tedious one) to show that for any $a, b \in \mathbb{R}$ we have

$$|\text{sgn}(a)|a|^{1/2} - \text{sgn}(b)|b|^{1/2}| \leq 2|a - b|.$$

If follows from this that $\|\Phi(f) - \Phi(g)\|_2^2 \leq 2\|f - g\|_1$ for all $f, g \in L^1(E)$. So $\Phi$ is a continuous mapping (to make $\|\Phi(f) - \Phi(g)\|_2 < \varepsilon$, consider $f, g \in L^1(E)$ such that
∥f − g∥₁ < δ = ε²/2). In fact, we see that Φ is uniformly continuous on L¹(E). Also Φ is one to one since Φ(f) = Φ(g) implies that f = g. Now Φ is onto L²(E) since its inverse is Φ⁻¹(f)(x) = sgn(f(x))|f(x)|² for f ∈ L²(E):

Φ(sgn(f(x))|f(x)|²) = sgn(sgn(f(x))|f(x)|)|sgn(f(x))|f(x)|²|¹/²

= sgn(f(x))|f(x)| = f(x).

Problem 11.38 gives that for all a, b ∈ ℝ, |sgn(a)|a|² − sgn(b)|b|²| ≤ 2|a − b|(|a| + |b|). Therefore ∥Φ⁻¹(f) − Φ⁻¹(g)∥₁ ≤ 2∥f − g∥²(∥f∥² + ∥g∥²). So Φ⁻¹ : L²(E) → L¹(E) is continuous: For given f ∈ L²(E), to make Φ⁻¹(f) − Φ⁻¹(g)∥₁ < ε, consider g ∈ L²(E) such that ∥f − g∥₂ < δ where δ = min{∥f∥₂, √ε/(√10∥f∥₂)}. We have for ∥f − g∥₂ < δ that ∥g∥₂ < ∥f∥₂ + δ < 2∥f∥₂ and so

∥Φ⁻¹(f) − Φ⁻¹(g)∥₁ ≤ 2∥f − g∥²(∥f∥² + ∥g∥²) < 2∥f − g∥²(∥f∥² + (2∥f∥₂)²)

= 10∥f − g∥₂∥f∥₂ < 10 \left( \frac{\sqrt{\varepsilon}}{\sqrt{10}\|f\|_2} \right)^2 \|f\|_2^2 = \varepsilon.

Therefore Φ : L¹(E) → L²(E) is a homeomorphism and L¹(E) is (topologically) isomorphic to L²(E). Of course, as Banach spaces, L¹(E) and L²(E) are quite different.

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