Chapter 12. Topological Spaces: Three Fundamental Theorems

Section 12.1. Urysohn's Lemma and the Tietze Extension Theorem

Note. In this section, we prove three major results. The two in the title of the section involve continuous real-valued functions. The third result is the Urysohn Metrization Theorem which concerns a certain type of metrizable topological space.

Urysohn's Lemma. Let A and B be disjoint closed subsets of a normal topological space $(X, \mathcal{T})$. Then for any closed bounded interval of real numbers $[a, b]$, there is a continuous real-valued function f defined on X that takes values in $[a, b]$, while $f = a$ on A and $f = b$ on B.

Note. The idea of Urysohn's Lemma is that, in a normal topological space, two closed sets can be separated and so we think of there as being some "space" between the closed sets and it is in this space (... "wiggle room") that we let the continuous function vary from a to b.

Definition. Let $(X, \mathcal{T})$ be a topological space and Λ a set of real numbers. A collection of open subsets of $\mathcal{T}$ indexed by Λ, $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, is *normally ascending* provided for any $\lambda_1, \lambda_2 \in \Lambda$, we have $\overline{\mathcal{O}}_{\lambda_1} \subseteq \mathcal{O}_{\lambda_2}$ if $\lambda_1 < \lambda_2$.

Example. Let f be a continuous real-valued function on $(X, \mathcal{T})$. Let Λ be any set of real numbers (in particular, Λ may not be countable) and define, for $\lambda \in \Lambda$,

$$\mathcal{O}_\lambda = \{x \in X \mid f(x) < \lambda\}.$$

Since f is continuous, then for $\lambda_1 < \lambda_2$ we have

$$\overline{\mathcal{O}}_{\lambda_1} \subseteq \{x \in X \mid f(x) \leq \lambda_1\} \subseteq \{x \in X \mid f(x) < \lambda_1\} = \mathcal{O}_{\lambda_2}$$

and therefore the collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is normally ascending.

Lemma 12.1. Let $(X, \mathcal{T})$ be a topological space. For Λ a dense subset of the open, bounded interval of real numbers (a, b), collection of open subsets of $\mathcal{T}$. Define the function $f : X \to \mathbb{R}$ by setting $f = b$ on $X \sim \cup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ and otherwise setting $f(x) = \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\}$. Then $f : X \to [a, b]$ is continuous on X.

Note. We leave the proof of Lemma 12.1 to Problems 12.9 and 12.10.

Lemma 12.2. Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and $\mathcal{U}$ a neighborhood of F. Then for any open, bounded interval (a, b), there is a dense subset Λ of (a, b) and a normally ascending collection of open subsets of X, $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, for which

$$F \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq \mathcal{U} \text{ for all } \lambda \in \Lambda.$$

Note. Now for the proof of Urysohn's Lemma using Lemmas 12.1 and 12.2.

Note. The following is a generalization of Urysohn's Lemma in the sense that it extends a function continuous on a closed subset of a topological space to a larger part of the space.

The Tietze Extension Theorem.

Let $(X, \mathcal{T})$ be a normal topological space, F a closed subset of X, and f a continuous real-valued function on F that takes values in the closed, bounded interval $[a, b]$. Then f has a continuous extension to all of X that also takes values in $[a, b]$.

Note. The Tietze Extension Theorem also holds when function f is unbounded. See Problem 12.8.

Note. We now apply Urysohn's Lemma to prove that the Urysohn Metrization Theorem from Section 11.3.

The Urysohn Metrization Theorem.

Let $(X, \mathcal{T})$ be a second countable topological space. Then $(X, \mathcal{T})$ is metrizable if and only if it is normal.

Note. Royden and Fitzpatrick make some claims in their proof of the Urysohn Metrization Theorem which are not entirely clear. So we include a detailed proof of Urysohn Metrization Theorem.

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