Chapter 14. Duality for Normed Linear Spaces

Note. In Section 8.1, we defined a linear functional on a normed linear space, a bounded linear functional, and the functional norm. In Proposition 8.1 (the proof is Exercise 8.2) it is shown that the collection of bounded linear functionals themselves form a normed linear space called the dual space of X, denoted X^*. In Chapters 14 and 15 we consider the mapping from $X \times X^* \to \mathbb{R}$ defined by $(x, \psi) \mapsto \psi(x)$ to "uncover the analytic, geometric, and topological properties of Banach spaces." The "departure point for this exploration" is the Hahn-Banach Theorem which is started and proved in Section 14.2 (Royden and Fitzpatrick, page 271).

Section 14.1. Linear Functionals, Bounded Linear Functionals, and Weak Topologies

Note. In this section we consider the linear space of all real valued linear functionals on linear space X (without requiring X to be named or the functionals to be bounded), denoted $X^\sharp$. We also consider a new topology on a normed linear space called the *weak topology* (the old topology which was induced by the norm we now may call the *strong topology*). For the deal X^* of normed linear space X, the weak topology is called the *weak-$*$ topology.*

Note. Recall that if Y and Z are subspaces of a linear space then $Y + Z$ is also a subspace of X (by Exercise 13.2) and that if $Y \cap Z = \{0\}$ then $Y + Z$ is denoted $T \oplus Z$ and is called the *direct sum* of Y and Z.

Definition. Let X be a linear space. The linear space of all linear real valued functions on X (whether the functionals are bounded or not) is denoted $X^\sharp$ (read "X sharp").

Lemma 14.1.A. Let X be a linear space and $\psi \in X^\sharp$, $\psi \neq 0$, and $x_0 \in X$ for which the direct sum $X = (\mathrm{Ker}(\psi)) \oplus \mathrm{span}\{x_0\}$, where $\mathrm{Ker}(\psi) = \{x \in X \mid \psi(x) = 0\}$.

Note. If $x_0 \in X$ and for $\psi \in X^\sharp$ where $\psi \neq 0$ we have $\psi(x_0) = c$, then for any $x \in X$ with $\psi(x) = c$ we have $\psi(x - x_0) = \psi(x) - \psi(x_0) = c - c = 0$. So $x = (x - x_0) + x_0$ and $x \in \mathrm{Ker}(\psi) + x_0$. Therefore $\psi^{-1}(c) = \{x \in X \mid \psi(x) = c\} = \mathrm{Ker}(\psi) + x_0$. By Lemma 14.1.A, if X is n-dimensional then $\mathrm{Ker}(\psi)$ is $(n-1)$-dimensional and in this case $\psi^{-1}(c)$ is the translate of the $(n-1)$-dimensional subspace (where x_0 is the translation vector and $\mathrm{Ker}(\psi)$ is the subspace).

Definition. Let X be a linear space and let X_0 be a linear subspace. If X_0 has the property that there is some $x - 0 \in X$, $x_0 \neq 0$, for which $X = X_0 + \mathrm{span}\{x_0\}$, then X_0 is a linear subspace of *codimension* 1 in X. A translate of a linear subspace of codimension 1 is called a *hyperplane*.

Proposition 14.1. A linear subspace X_0 of a linear space X is of codimension 1 if and only if $X_0 = \mathrm{Ker}(\psi)$ for some nonzero $\psi \in X^\sharp$.

Note. The following is an "extension theorem" in the same sense as is the "Tietze Extension Theorem" from Section 12.1.

Proposition 14.2. Let Y be a linear subspace of a linear space X. Then each linear functional on Y is an extension to a linear functional on all of X. In particular, for each nonzero $x \in X$ there is a $\psi \in X^\sharp$ for which $\psi(x) \neq 0$.

Note. Since X^* is the (normed) linear space of all *bounded* linear functionals on X, then X^* is a subset of $X^\sharp$, the linear space of all linear functionals on X. In Exercise 14.3 it is shown that if X is a finite dimensional normed linear space that every linear functional on X is continuous and hence (by Theorem 13.1) bounded. So in a finite dimensional normed linear space, $X^* = X^\sharp$. In fact, this property can be used to classify a normed linear space as finite or infinite dimensional (similar to Riesz's Theorem of Section 13.3 which classified these spaces by considering the compactness of the closed unit ball), as we'll see in Propostion 14.3.

Definition. Let X be a linear space. A set $\mathcal{B} \subset X$ is a *Hamel basis* if every vector in X is expressible as a unique finite linear combination of vectors in $\mathcal{B}$.

Note. In Exercise 14.16 it is shown that every linear space has a Hamel basis using Zorn's Lemma. The fact that Zorn's Lemma (which is equivalent to the Axiom of Choice) is needed for this proof means that we have no idea what is in a Hamel basis, just as in Section 2.6 when we "constructed" a nonmeasurable set P using the Axiom of Choice, we had no idea what was in set P. This makes it impractical to use a Hamel basis in any area of applied math. Though we do apply the existence of a Hamel basis in the proof of the following.

Proposition 14.3. Let X be a normed linear space X is finite dimensional if and only if $X^\sharp = X^*$.

Note. The following relates elements of $X^\sharp$ to subsets of $X^\sharp$ in terms of linear combinations and will be used when studying properties of weak topologies.

Proposition 14.4. Let X be a linear space, let $\psi \in X^\sharp$ and $\{\psi_i\}_{i=1}^n \subset X^\sharp$. Then ψ is a linear combination of $\{\psi_i\}_{i=1}^n$ if and only if $\cap_{i=1}^n \mathrm{Ker}(\psi_i) \subset \mathrm{Ker}(\psi)$.

Recall. In Section 11.4 we defined topology $\mathcal{T}_1$ on set X to be *stronger* than topology $\mathcal{T}_2$ on set X (or $\mathcal{T}_2$ is *weaker* than $\mathcal{T}_1$) if $\mathcal{T}_2 \subset \mathcal{T}_2$. also, for X any set and a collection of mappings $\mathcal{F} = \{f : X \to \mathbb{R}\}$, the weakest topology for X that contains the collection of sets $\{f^{-1}(\mathcal{O}) \mid f \in \mathcal{F}, \mathcal{O}$ is open in $\mathbb{R}\}$ is the *weak topology* for X induced by $\mathcal{F}$. By Proposition 11.13 the weak topology induced by $\mathcal{F}$ has the fewest number of sets among the topologies on X for which each mapping of $\mathcal{F}$ is continuous. A base for the $\mathcal{F}$-weak topology on X at $x \in x$ is given by sets of the form

$$\mathcal{N}_{\varepsilon,f_1,f_2,\ldots,f_n}(x) = \{x' \in X \mid |f_j(x') - f(x)| < \varepsilon \text{ for } 1 \leq k \leq n\}$$

where $\varepsilon > 0$ and $\{f_k\}_{k=1}^n$ is a finite subcollection of $\mathcal{F}$.

Note. A sequence $\{x_n\} \subset X$ converges to $x \in X$ with respect to the $\mathcal{F}$-weak topology if an only if $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in \mathcal{F}$ since each f is continuous in the $\mathcal{F}$-weak topology.

Definition. A function on X that is continuous with respect to the $\mathcal{F}$-weak topology is *$\mathcal{F}$-weakly continuous*. A set open in the $\mathcal{F}$-weak topology is *$\mathcal{F}$-weakly open*. We similarly have *$\mathcal{F}$-weakly closed* sets and *$\mathcal{F}$-weakly compact* sets.

Proposition 14.5. Let X be a linear space and W a subspace of $X^\sharp$. Then a linear functional $\psi : X \to \mathbb{R}$ is E-weakly continuous if and only if it belongs to W.

Note. Any collection $\mathcal{F}$ of real valued functions on a set X determines the $\mathcal{F}$-weak topology on X. This yields a special topology on X when $\mathcal{F} = X^*$ (the bounded linear functionals on X).

Definition. Let X be a normed linear space. The weak topology induced on X by the dual space X^* is the *weak topology* on X.

Note. By Theorem 13.1, a linear functional mapping X to $\mathbb{R}$ is bounded (i.e., in X^*) if and only if it is continuous (that is, continuous with respect to the topology on X which is induced by the norm on X; $\mathbb{R}$ had the usual Euclidean topology given by the absolute value metric). So the weak topology on X induced by the norm on X (which we now call the *strong topology*). In Exercise 14.6 it is shown that the strong and weak topologies on X coincide if and only if X is finite dimensional.

Note. A base for the weak topology on X at $x \in X$ is given by sets of the form

$$\mathcal{N}_{\varepsilon,\psi_1,\psi_2,\ldots,\psi_n}(x) = \{x' \in X \mid |\psi_k(x') - \psi_k(x)| < \varepsilon \text{ for } 1 \leq k \leq n\}$$

where $\varepsilon > 0$ and $\{\psi_k\}_{k=1}^n$ is a finite subcollection of X^*.

Note. As opposed to the terminology "X^*-weakly continuous/convergent/open/closed" we just say "weakly continuous/...." Notice that a sequence $\{x_n\}$ in X converges weakly to $x \in X$ if and only if $\lim_{n\to\infty} \psi(x_n) = \psi(x)$ for all $\psi \in X^*$. For weak convergence, we write $\{x_n\} \rightharpoonup x$.

Definition. Let X be a normed linear space. For $x \in X$ define functional $J(x) : X^* \to \mathbb{R}$ by $J(x)[\psi] = \psi(x)$ for all $\psi \in X^*$. This is the *evaluation functional* (at x).

Lemma 14.1.B. The evaluation functional $J(x)$ is linear and bounded. That is, $J(x) \in (X^*)^*$.

Note. We can consider J itself as a function of $x \in X$ to $J(x) \in (X^*)^*$. So $J : X \to (X^*)^*$. Now J is linear since

$$J(\alpha_1 x_1 + \alpha_2 x_2)[\psi] = \psi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \psi(x_1) + \alpha_2 \psi(x_2) = \alpha_1 J(x_1)[\psi] + \alpha_2 J(x_2)[\psi]$$

for all $\psi \in X^*$, so $J(\alpha_1 x_2 + \alpha_2 x_2) = \alpha_1 J(x_0) + \alpha_2 J(x_2)$. Therefore $J(X)$ is a linear subspace of $(X^*)^*$ (by definition of "linear subspace"; $J(X)$ is clearly closed under linear combinations). Since $J(X)$ is a collection of real valued functions on X^*, then we can consider the "$J(X)$-weak topology" on X^* induced by the set $J(X)$.

Definition. Let X be a normed linear space. The weak topology on X^* induced by $J(X) \subset (X^*)^*$, where $J(x) : X^* \to \mathbb{R}$ defined as $J(x)[\psi] = \psi(x)$ for all $\psi \in X^*$, is the *weak-$*$ topology* on X^*.

Note. A base for the weak-$*$ topology on X^* at $\psi \in X^*$ is given by sets of the form

$$\mathcal{N}_{\varepsilon, x_1, x_2, \ldots, x_n}(\psi)\{\psi' \in X^* \mid |\psi'(x_k) - \psi(x_k)| < \varepsilon \text{ for } 1 \leq k \leq n\}$$

where $\varepsilon > 0$ and $\{x_k\}_{k=1}^n$ is a finite subset of X. We, similar to above, refer to weak-$*$ continuous.convergent.open/closed. Notice that a sequence $\{\psi_n\}$ in X^* is weak-$*$ convergent to $\psi \in X^*$ if and only if $\lim_{n\to\infty} \psi(x_n) = \psi(x)$ for all $x \in X$.

Note. For normed linear space X, we have the following relationships on the topologies on X^*:

$$\text{weak-}* \text{ topology on } X^* \subset \text{weak topology on } X^* \subset \text{strong topology on } X^*.$$

Definition. Let X be a normed linear space. The linear operator $J : X \to (X^*)^*$, defined by $J(x)[\psi] = \psi(x)$ for all $x \in X$ and $\psi \in X^*$, is the *natural embedding* of X into $(X^*)^*$. The space X is *reflexive* provided $J(X) = (X^*)^*$. We denote $(X^*)^*$ as X^{**} and call it the *bidual* of X (or the *double dual*).

Note. We now classify reflexive spaces in terms of their topologies.

Proposition 14.6. A normed linear space X is reflexive if and only if the weak and weak-$*$ topologies are the same.

Note. The term "embedding" implies a one to one mapping, but we have not shown that $J : X \to X^{**}$ is one to one. We have not even established that there are nonzero bounded linear functionals on a general normed linear space (that is, we do not know that $|X^*| > 1$). In the next section, we prove the Hahn-Banach Theorem which will allow us to prove the existence of bounded linear functionals from a subspace (Theorem 14.7). We will also show that $J : X \to X^{**}$ is an isometry (in Corollary 14.9) and so is one to one.

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