Chapter 17. General Measure Spaces: Their Properties and Construction

Note. In Chapter 2, we defined Lebesgue measure on the σ-algebra $\mathcal{M}$, and used open sets (open intervals, specifically) in the definition. In this chapter, we give an abstract definition of a measure on a σ-algebra of some point set. The technique is analogous to the definition of a topology—the sets which are open are, by definition, the sets in the topology. Hence, the sets which are measurable are, by definition, the sets in the σ-algebra. We will still, however, have some analogies with outer measure and the Carathéodory splitting condition.

Section 17.1. Measures and Measurable Sets

Note. In this section, we define a measure space and show parallels between this new setting and the results of Chapter 2.

Definition. A *measurable space* is an ordered pair $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. Set $E \subset X$ is *measurable* if $E \in \mathcal{M}$. A *measure* μ on a measurable space $(X, \mathcal{M})$ is an extended real-valued nonnegative set function $\mu : \mathcal{M} \to [0, \infty]$ for which $\mu(\varnothing) = 0$ and which is *countably additive* in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\mu\left(\dot{\bigcup_{k=1}^{\infty}} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

A *measure space* $(X, \mathcal{M}, \mu)$ is a measurable space $(X, \mathcal{M})$ together with a measure μ on $\mathcal{M}$.

Note. We denote the power set of X, motivated by properties of cardinal numbers, as $\mathcal{P}(X) = 2^X$.

Example. For any set X, we can take $\mathcal{M} = 2^X$ and define η on $\mathcal{M}$ as $\eta(A) = |A|$ for finite set A and $\eta(B) = \infty$ for infinite set B. Then $(X, \mathcal{M}, \eta)$ is a measure space. η is called the *counting measure* on X.

Example. For X uncountable and $\mathcal{C}$ the collection of subsets of X that are either countable or the complement of a countable set. We claim that $\mathcal{C}$ is a σ-algebra and define μ as $\mu(A) = 0$ for A countable and $\mu(B) = 1$ for B where $X \setminus B$ is countable. (Notice that there are not two disjoint sets of measure 1, so countable additivity is not an issue.) Then $(X, \mathcal{C}, \mu)$ is a measure space.

Proposition 17.1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets,

$$\mu\left(\dot{\bigcup_{k=1}^{n}} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

That is, μ is *finite additive*.

(ii) If A and B are measurable sets and $A \subseteq B$, then $\mu(A) \leq \mu(B)$. That is, μ is *monotone*.

(iii) If A and B are measurable sets, $A \subseteq B$, and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. This is the *excision principle*.

(iv) For any countable collection $\{E_k\}_{k=1}^\infty$ of measurable sets that covers a measurable set E,

$$\mu(E) \leq \sum_{k=1}^\infty \mu(E_k).$$

This is called *countable monotonicity*.

Note. The following is analogous to Theorem 2.15. In fact, the proof of this new result is identical to the proof of Theorem 2.15.

Proposition 17.2. Continuity of Measure.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) If $\{A_k\}_{k=1}^\infty$ is an ascending sequence of measurable sets (i.e., $A_k \subseteq A_{k+1}$), then

$$\mu\left(\bigcup_{k=1}^\infty A_k\right) = \mu\left(\lim_{k\to\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

(ii) If $\{B_k\}_{k=1}^\infty$ is a descending sequence of measurable sets (i.e., $B_k \supseteq B_{k+1}$) for which $\mu(B_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^\infty B_k\right) = \mu\left(\lim_{k\to\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k).$$

Definition. For a measure space $(X, \mathcal{M}, \mu)$ and measurable $E \subseteq X$, a property holds *almost everywhere* on E (denoted "a.e. on E") if the property holds on $E \setminus E_0$ where E_0 is measurable, $E_0 \subseteq E$, and $\mu(E_0) = 0$.

Note. The Borel-Cantelli Lemma from Section 2.5 also holds in measure spaces.

The Borel-Cantelli Lemma. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\displaystyle\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all $x \in X$ belong to at most a finite number of the E_k's.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is *finite* provided $\mu(X) < \infty$. The measure is σ*-finite* if X is the union of a countable collection of measurable sets, each of which has finite measure. $E \in \mathcal{M}$ is of *finite measure* if $\mu(E) < \infty$. $E \in \mathcal{M}$ is σ*-finite* if E is the union of a countable collection of measurable sets, each of which has finite measure.

Example. Lebesgue measure on $[0, 1]$ is a finite measure. Lebesgue measure on $\mathbb{R}$ is a σ-finite measure. The counting measure on an uncountable set is not σ-finite.

Note. Many of the properties of Lebesgue measure and Lebesgue integration hold for arbitrary measures which are σ-finite, and the property of σ-finite is often necessary for the properties to hold.

Note. By Proposition 2.4, any subset of $\mathbb{R}$ of outer measure zero is measurable. In particular, any subset of a set of real numbers of outer measure zero is measurable. However, this is not necessarily a property of a measure space. Yet we still desire this property.

Definition. A measure space $(X, \mathcal{M}, \mu)$ is *complete* if $\mathcal{M}$ contains all subsets of sets of measure zero.

Example. Lebesgue measure on the σ-algebra of Lebesgue measurable sets is a complete measure by Proposition 2.4. Lebesgue measure on the Borel sets is not a complete measure space since there are subsets of the Cantor set (which is of measure zero) which are not Borel. This is argued on page 52, but can also be shown by a cardinality argument: $|\mathcal{B}| = \aleph_1$ and $|\mathcal{P}(C)| = \aleph_2$.

Proposition 17.3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Define

$$\mathcal{M}_0 = \{E \subseteq X \mid E = A \cup B, B \in \mathcal{M}, A \subseteq C \text{ for some } C \in \mathcal{M} \text{ with } \mu(C) = 0\}.$$

Define $\mu_0(E) = \mu(B)$ for all $E \in \mathcal{M}_0$. Then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ to $\mathcal{M}_0$, and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space, called the *completion* of $(X, \mathcal{M}, \mu)$.

Proof. Problem 17.9.

Problem 17.5. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference, $E_1 \triangle E_2$, of two subsets E_1 and E_2 of X is defined as $E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

(i) Show that if $E_1, E_2 \in \mathcal{M}$ and $\mu(E_1 \triangle E_2) = 0$, then $\mu(E_1) = \mu(E_2)$.

(ii) Show that if μ is complete, $E_1 \in \mathcal{M}$ and $E_2 \setminus E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ if $\mu(E_1 \triangle E_2) = 0$.