Section 17.2. Signed Measures: The Hahn and Jordan Decompositions

Note. If measure space $(X, \mathcal{M})$ admits measures μ_1 and μ_2, then for any $\alpha, \beta \in \mathbb{R}$ where $\alpha \geq 0, \beta \geq 0$, $\mu_3 = \alpha\mu_1 + \beta\mu_2$ is a measure on $(X, \mathcal{M})$. Inspired by this observation, we attempt to deal with *any* linear combination of measures—in particular, a difference of two measures $\mu_1 - \mu_2$. This leads us to consider signed measures (we must avoid any $\infty - \infty$ situations, though).

Definition. A *signed measure* ν ("nu") on the measure space $(X, \mathcal{M})$ is an extended real-valued set function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that:

(i) ν assumes at most one of the values $-\infty$, $+\infty$.

(ii) $\nu(\varnothing) = 0$.

(iii) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable sets,

$$\nu\left(\dot{\bigcup_{k=1}^{\infty}} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the series on the right converges absolutely if $\nu(\cup_{k=1}^{\infty} E_k)$ is finite.

Note. The difference of two measures, one of which only assumes finite values, is a signed measure. In fact, we'll see that every signed measure is the difference of two such measures (in the Jordan Decomposition Theorem).

Definition. Let ν be a signed measure on $(X, \mathcal{M})$. Then the set $A \subset X$ is *positive* with respect to ν if $A \in \mathcal{M}$ and for each $E \subset A$ with $E \in \mathcal{M}$ we have $\nu(E) \geq 0$. Set $B \subset X$ is *negative* with respect to ν if $B \in \mathcal{M}$ and for each $E \subset B$ with $B \in \mathcal{M}$ we have $\nu(B) \leq 0$. Set $C \in \mathcal{M}$ is *null* with respect to ν is $C \in \mathcal{M}$ and for each $E \subset C$ with $E \in \mathcal{M}$ we have $\nu(E) = 0$.

Note. A null set is both positive and negative.

Note. If $A \subset X$ is positive, then signed measure ν restricted to the measurable subsets of A is a measure. If $B \subset X$ is negative, then signed measure $-\nu$ restricted to the measurable subsets of B is a measure.

Note. There is a difference in a null set and a measure zero set. A null set must be of measure zero, but a measure zero set may be the disjoint union of two sets, one the (nonzero) negative measure of the other.

Lemma 1. Let ν be a signed measure on $(X, \mathcal{M})$, $A, B \in \mathcal{M}$, $|\nu(B)| < \infty$, and $A \subset B$. Then $|\nu(A)| < \infty$. (NOTE: We do not have monotonicity under a signed measure, but we do have this monotonicity-like result.)

Proof. We have $A \cup (B \setminus A) = B$, so $\nu(A \cup (B \setminus A)) = \nu(B)$ and $\nu(A) + \nu(B \setminus A) = \nu(B)$. Since $\nu(B)$ is finite, then $\nu(A) + \nu(B \setminus A)$ is finite and so both $\nu(A)$ and $\nu(B \setminus A)$ are finite ($\infty - \infty$ is not allowed in a signed measure). ∎

Proposition 17.4. Let ν be a signed measure on $(X, \mathcal{M})$. Then the union of a countable collection of positive sets is positive.

Hahn's Lemma. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$, that is positive and of positive measure.

Note. We now use a signed measure on $(X, \mathcal{M})$ to decompose the point set X into a disjoint union of a positive set and a negative set. We then take the signed measure itself and decompose it into positive and negative parts.

Definition. Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. If $X = A \cup B$ where A is a positive set, B is a negative set, and $A \cap B = \varnothing$, then the pair $\{A, B\}$ is a *Hahn-decomposition* of X.

The Hahn Decomposition Theorem. Let ν be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of X.

Note. A Hahn decomposition $\{A, B\}$ is not necessarily unique since we can extract a null set from A (or B) and union it with B (or A) to get a different Hahn decomposition of X. However, in a sense, the Hahn decomposition is unique "modulo" null sets. This is spelled out more clearly in the following result.

Lemma 2. Let ν be a signed measure on $(X, \mathcal{M})$. If $\{A, B\}$ and $\{A', B'\}$ are both Hahn decompositions of X, then A and A' differ only be a null set, and B and B' differ only by a null set. That is, $A \triangle A' = (A \setminus A') \cup (A' \setminus A)$ and $B \triangle B' = (B \setminus B') \cup (B' \setminus B)$ are null sets.

Proof. Consider $A \cap B' \subset A$. Since A is positive, then $\nu(A \cap B') \geq 0$ and since B' is negative, $\nu(A \cap B') \leq 0$. So $\nu(A \cap B') = 0$. That is, $\nu(A \setminus A') = 0$. Similarly, $A' \cap B = A' \setminus A$, $B \cap A' = B \setminus B'$, and $B' \cap A = B' \setminus B$ are also null sets and the result follows by additivity. ∎

Definition. Two measures ν_1 and ν_2 on $(X, \mathcal{M})$ are *mutually singular*, denoted $\nu_1 \perp \nu_2$, if there are disjoint measurable sets A and B with $X = A \cup B$ for which $\nu_1(A) = \nu_2(B) = 0$.

The Jordan Decomposition Theorem.

Let ν be a signed measure on $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Definition. The decomposition of signed measure ν on measure space $(X, \mathcal{M})$ into the difference of two (nonnegative) measures given in the Jordan Decomposition Theorem is called the *Jordan decomposition* of ν.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable over $\mathbb{R}$ (i.e., $\int_{\mathbb{R}} f < \infty$). For Lebesgue measurable $E \subset \mathbb{R}$, define $\nu(E) = \int_E f$. Since Lebesgue integration is countably additive (Theorem 4.20), then ν is a signed measure on $(\mathbb{R}, \mathcal{L})$ where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets. Define

$$A = \{x \in \mathbb{R} \mid f(x) \geq 0\} \text{ and } B = \{x \in \mathbb{R} \mid f(x) < 0\}.$$

For $E \in \mathcal{L}$, define $v^+(E) = \int_{A \cap E} f$ and $v^-(E) = -\int_{B \cap E} f$. Then $\{A, B\}$ is a Hahn decomposition of $\mathbb{R}$ with respect to ν and $v = v^+ - v^-$ is the Jordan decomposition of ν.

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