Section 17.3. The Carathéodory Measure Induced by an Outer Measure

Note. In this section, we consider set functions μ^* defined on $\mathcal{P}(X) = 2^X$ (a new notation for the power set) which satisfies certain properties and take on values in $[0,\infty]$ (so we are leaving signed measure behind). If μ^* satisfies these certain properties, it is called an outer measure. We again consider the Carathéodory splitting condition and define a measure μ based on μ^*. Therefore, we are totally mimicking the development of Lebesgue measure in this abstract setting. What makes this setting "abstract" is that we don't have a very hands-on feel for μ or μ^* (unlike in the development of Lebesgue measure, which was all ultimately based on open intervals).

Definition. A set function $\mu : \mathcal{S} \to [0,\infty]$ defined on a collection $\mathcal{S}$ of subsets of a set X is *countably monotone* provided that whenever $E \subset \cup_{k=1}^{\infty} E_k$ where E and each E_k are in $\mathcal{S}$, then $\mu(E) \le \sum_{k=1}^{\infty} \mu(E_k)$.

Note. If $\varnothing \in \mathcal{S}$ and $\mu(\varnothing) = 0$, then μ is finitely monotone: $\mu(E) \le \sum_{k=1}^{n} \mu(E_k)$, since we simply take $E_k = \varnothing$ for $k > n$ and use countable monotonicity.

Definition. A set function is *monotone* on $\mathcal{S}$ if for each $A, B \in \mathcal{S}$ with $A \subset B$, we have $\mu(A) \le \mu(B)$.

Definition. A set function $\mu^* : 2^X \to [0, \infty]$ is an *outer measure* if $\mu^*(\varnothing) = 0$ and μ^* is countably monotone.

Note. As with Lebesgue measure, we use the Carathéodory splitting condition to define "measurable." Also like Lebesgue measure, this will give us the equipment to prove the usual properties of a measure.

Definition. For an outer measure $\mu^* : 2^X \to [0, \infty]$, we call a subset E of X *measurable* (with respect to μ^*) if for every subset A of X we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Note. Again, this is an "abstract setting" since we still have not actually defined any outer measure, but have merely *imposed* an outer measure on 2^X. We define outer measure on page 350, though even then it is in terms of a set function which is not specifically defined.

Note. Trivially, if E is measurable then $E^c = X \setminus E$ is measurable. Since μ^* is finitely monotone, to show $E \subset X$ is measurable we need only show

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \text{ for all } A \subset X, \mu^*(A) < \infty.$$

Proposition 17.5. The union of a finite collection of measurable sets is measurable.

Note. At this stage, we know that the measurable sets form an algebra of sets.

Proposition 17.6. Let $A \subset X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then

$$\mu^*\left(A \cap [\cup_{k=1}^n E_k]\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

That is, μ^* is finite additive on the measurable sets (which follows with $A = X$).

Proposition 17.7. The union of a countable collection of measurable sets is measurable.

Note. We now have that the measurable sets (with respect to μ^*) form a σ-algebra. However, we have not yet shown that we have a *measure* since we have not shown countable additivity (see the definition on page 338). We do so in a restricted sense in the following.

Theorem 17.8. Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.

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