Section 17.4. The Construction of Outer Measures

Note. In this section, we replace the open intervals in $\mathbb{R}$ with some subsets of X and define an outer measure using these subsets. The outer measure then induces a measure on some σ-algebra on X using the Carathéodory condition.

Theorem 17.9. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\varnothing) = 0$ and for $E \subset X$, $E \neq \varnothing$, define

$$\mu^*(E) = \inf \left(\sum_{k=1}^{\infty} \mu(E_k) \right),$$

where the infimum is taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E. Then the set function $\mu^* : 2^X \to [0, \infty]$ is an outer measure (called the *outer measure induced* by μ).

Definition. Let $\mathcal{S}$ be a collection of subsets of X, $\mu : \mathcal{S} \to [0, \infty]$ a set function, and μ^* the outer measure induced by μ. The measure $\overline{\mu}$ that is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets is the *Carathéodory measure induced* by μ.

Note. Continuing to parallel the development of Lebesgue measure, we use the sets in $\mathcal{S}$ to define collections of sets analogous to the F_σ, G_δ, $F_{\sigma\delta}$, $G_{\delta\sigma}$, etc. sets.

Definition. For collection $\mathcal{S}$ of subsets of X, define

$$S_\sigma = \left\{ A \subset X \,\middle|\, A = \bigcup_{k=1}^{\infty} S_k, S_k \in \mathcal{S} \right\}$$

and

$$S_{\sigma\delta} = \left\{ B \subset X \,\middle|\, B = \bigcap_{j=1}^{\infty} A_j, A_j \in S_\sigma \right\}.$$

Note. The following result is similar to Theorem 2.11, with G_δ sets replaced with $S_{\sigma\delta}$ sets. (Recall that we started with open intervals and then can generate open sets by taking countable unions of open intervals. So the development of Lebesgue integration is similar to what we are doing now, with set $\mathcal{S}$ being the set of open intervals for Lebesgue).

Proposition 17.10. Let $\mu : \mathcal{S} \to [0, \infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and let $\overline{\mu} : \mathcal{M} \to [0, \infty]$ be the Carathéodory measure induced by μ. Let $E \subset X$ satisfy $\mu(E) < \infty$. Then there is $A \subset X$ for which $A \in \mathcal{S}_{\sigma\delta}$, $E \subset A$, and $\mu^*(E) = \mu^*(A)$. Furthermore, if $E \in \mathcal{M}$ and $\mathcal{S} \subset \mathcal{M}$, then $A \in \mathcal{M}$ and $\overline{\mu}(A \setminus E) = 0$.

Revised: 3/29/2017