# Section 17.5. The Carathéodory-Hahn Theorem: The Extension of a Premeasure to a Measure

**Note.** We have $\mu$ defined on $\mathcal{S}$ and use $\mu$ to define $\mu^*$ on $X$. We then restrict $\mu^*$ to a subset of $X$ on which $\mu^*$ is a measure (denoted $\overline{\mu}$). Unlike with Lebesgue measure where the elements of $\mathcal{S}$ are measurable sets themselves (and $\mathcal{S}$ is the set of open intervals), we *may not* have $\overline{\mu}$ defined on $\mathcal{S}$. In this section, we look for conditions on $\mu : \mathcal{S} \to [0, \infty]$ which imply the measurability of the elements of $\mathcal{S}$. Under these conditions, $\overline{\mu}$ is then an extension of $\mu$ from $\mathcal{S}$ to $\mathcal{M}$ (the $\sigma$-algebra of measurable sets).

**Definition.** A set function $\mu : \mathcal{S} \to [0, \infty]$ is *finitely additive* if whenever $\{E_k\}_{k=1}^{\infty}$ is a finite disjoint collection of sets in $\mathcal{S}$ and $\cup_{k=1}^{\infty} E_k \in \mathcal{S}$, we have

$$\mu\left(\dot{\bigcup_{k=1}^{n}} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

**Note.** We have previously defined *finitely monotone* for set functions and Proposition 17.6 gives finite additivity for an *outer measure* on $\mathcal{M}$, but this is the first time we have discussed a finitely additive *set function*.

**Proposition 17.11.** Let $\mathcal{S}$ be a collection of subsets of $X$ and $\mu : \mathcal{S} \to [0, \infty]$ a set function. In order that the Carathéodory measure $\overline{\mu}$ induced by $\mu$ be an extension of $\mu$ (that is, $\overline{\mu} = \mu$ on $\mathcal{S}$) it is necessary that $\mu$ be both finitely additive and countably monotone and, if $\varnothing \in \mathcal{S}$, then $\mu(\varnothing) = 0$.

**Note.** We want to create measures using set functions, outer measure, and the Carathéodory condition. We want, as with Lebesgue measure, the Carathéodory measure to extend the set function, so we want the conditions of Proposition 17.11 to be satisfied. So, in light of Proposition 17.11, we have the following definitions.

**Definition.** Let $\mathcal{S}$ be a collection of subsets of $X$ and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Then $\mu$ is a *premeasure* if $\mu$ is both finitely additive and countably monotone and, if $\varnothing \in \mathcal{S}$, then $\mu(\varnothing) = 0$.

**Note.** The condition of a set function being a premeasure is necessary (by Proposition 17.11) but not sufficient to guarantee that $\overline{\mu}$ is an extension of $\mu$, as shown in Problems 17.25 and 17.26. It turns out that we need more "algebraic" structure on the sets in $\mathcal{S}$.

**Definition.** A collection $\mathcal{S}$ of subsets of $X$ is said to be *closed with respect to the formation of relative complements* provided that for any $A, B \in \mathcal{S}$, the relative complement $A \setminus B \in \mathcal{S}$. The collection $\mathcal{S}$ is said to be *closed with respect to the formation of finite intersections* provided for any $A, B \in \mathcal{S}$, the intersection $A \cap B \in \mathcal{S}$.

**Note.** Closure with respect to relative complements implies closure with respect to finite intersections since $A \cap B = A \setminus (A \setminus B)$. Closure with respect to relative complements also implies $\varnothing \in \mathcal{S}$ since $\varnothing = A \setminus A$.

**Theorem 17.12.** Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a nonempty collection $\mathcal{S}$ of subsets of $X$ that is closed with respect to the formation of relative complements. Then the Carathéodory measure $\overline{\mu} : \mathcal{M} \to [0, \infty]$ induced by $\mu$ is an extension of $\mu$ called the *Carathéodory extension* of $\mu$.

**Note.** As observed above, we need a certain amount of algebraic structure on $\mathcal{S}$ (in terms of relative complements and so forth) before we can conclude that $\overline{\mu}$ is an extension of $\mu$. This is the motivation for the following definition.

**Definition.** A nonempty collection $\mathcal{S}$ of subsets of $X$ is a *semiring* if for all $A, B \in \mathcal{S}$, we have $A \cap B \in \mathcal{S}$ and there is a finite disjoint collection $\{C_k\}_{k=1}^n$ of sets in $\mathcal{S}$ for which $A \setminus B = \dot\cup_{k=1}^n C_k$.

**Note.** The following result concerning semiring $\mathcal{S}$ is a preamble to our main result for this section (and requires a fairly lengthy proof).

**Proposition 17.13.** Let $\mathcal{S}$ be a semiring of subsets of a set $X$. Define $\mathcal{S}'$ to be the collection of unions of finite disjoint collections of sets in $\mathcal{S}$. Then $\mathcal{S}'$ is closed with respect to the formation of relative complements. Furthermore, any premeasure on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$.

**Definition.** Let $\mathcal{S}$ be a collection of subsets of $X$. A set function $\mu : \mathcal{S} \to [0, \infty]$ is *$\sigma$-finite* if $X = \cup_{k=1}^{\infty} S_k$ where $S_k \in \mathcal{S}$ and $\mu(S_k) < \infty$ for each $k \in \mathbb{N}$.

**The Carathéodory-Hahn Theorem.**

Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a semiring $\mathcal{S}$ of subsets of $X$. Then the Carathéodory measure $\overline{\mu}$ induced by $\mu$ is an extension of $\mu$. Furthermore, if $\mu$ is $\sigma$-finite, then so is $\overline{\mu}$ and $\overline{\mu}$ is the unique measure on the $\sigma$-algebra of $\mu^*$-measurable sets that extends $\mu$.

**Corollary 17.14.** Let $\mathcal{S}$ be a semiring of subsets of a set $X$ and $\mathcal{B}$ the smallest $\sigma$-algebra of subsets of $X$ that contain $\mathcal{S}$. Then two $\sigma$-finite measures on $\mathcal{B}$ are equal if and only if they agree on sets in $\mathcal{S}$.

**Note.** The assumption of $\sigma$-finite in the Carathéodory-Hahn Theorem is necessary for the uniqueness claim, as shown by example in Problem 17.32.

**Note.** The application of the Carathéodory-Hahn Theorem of most importance is the introduction of product measures (in Sections 20.1 and 20.2). As a first example, the bounded intervals of real numbers form a semiring (see Problem 17.33i) and the set function which associated the length of an interval with the interval is a premeasure. So by the Carathéodory-Hahn Theorem, Lebesgue measure (which *is* the Carathéodory measure induced by interval length) is the unique $\sigma$-finite extension of length on the $\sigma$-algebra of Lebesgue measurable sets.

**Note.** We conclude this chapter with a few set-theoretic definitions.

**Definition.** Let $\mathcal{S}$ be a collection of subsets of $X$. Then $\mathcal{S}$ is a *ring* of sets if it is closed with respect to finite unions and the formation of relative complements (and therefore by DeMorgan, closed with respect to finite intersections). A ring that contains $X$ is an *algebra* (again). A semiring that contains $X$ is a *semialgebra*.

*Revised: 4/18/2019*